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A comparison of parametric models for mortality graduation. Application to mortality data for the Valencia Region (Spain)

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Abstract

The parametric graduation of mortality data has as its objective the satisfactory estimation of the death rates based on mortality data but using an age-dependent function whose parameters are adjusted from the crude rates obtainable directly from the data. This paper proposes a revision of the most commonly used parametric methods and compares the results obtained with each of them when they are applied to the mortality data for the Valencia Region. As a result of the comparison, we conclude that the Gompertz-Makeham functions estimated by means of generalized linear models lead to the best results. Our working method is of additional interest for being applicable to mortality data for a wide range of ages from any geographical conditions, allowing us to select the most appropriate life table for the case in hand.

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Keywords: Gompertz-Makeham functions, Heligman and Pollard's laws, parametric graduation.

1 Introduction

Historically, Actuarial Science has worked with the mortality data of a population. The first step, and perhaps one of the fundamental ways in which statistics plays a part, is the graduation of mortality data. We define graduation (Haberman and Renshaw, 1996) as

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the set of principles and methods by which the observed (or crude) probabilities are fitted to provide a smooth basis for making practical inferences and calculations of premiums and reserves. Graduation is necessary (London, 1985) because the sequence of crude death probabilities generally presents brusque changes, which do not correspond the plausible hypothesis that the probabilities of death for two consecutive ages should be very close.

The graduation methods suggested in the literature, and used in practice, can be classified into two fundamental types: parametric and non-parametric, depending on whether they adjust the data to a function or simply achieve smoothness. Within the first type are the now classic Gompertz (1825) and Makeham (1860) models, used especially for advanced age groups: the former postulate that the force of mortality would grow exponentially with age, and the second adds a constant, an age independent component, to the exponential growth. These authors' proposals gave good results for data from the late 19th and early 20th centuries. Over time a mortality pattern evolved with an increase in mortality among the young and a relative hump among the middle-aged, such that it was difficult to obtain a good graduation with the Makeham formula, which in turn led to the introduction of new models known as the Heligman and Pollard laws (Heligman and Pollard, 1980). The Gompertz-Makeham function described in Forfar *et al.* (1988) generalizes the original models proposed by Gompertz and Makeham. In Renshaw (1991) and Renshaw *et al.* (1997), generalized linear and non-linear models are used for adjusting these functions. An example of non-parametric graduation by means of kernel smoothing can be found in Gavin *et al.* (1993, 1995).

The objective of this paper is to revise and compare different parametric graduation models by applying them to real mortality data for the Valencia Region, on the Spanish Mediterranean coast. The paper is organized as follows. In Section 2 we present the parametric graduation models: the methodology developed by the Continuous Mortality Investigation (CMI) Bureau and its extension to generalized linear models, and the so-called Heligman and Pollard model. Section 3 is devoted to obtaining crude estimations of the probability of death in the Valencia Region for the period 1999-2001. We apply the different graduation methods to these estimations, commenting on their advantages and disadvantages, as well as on their suitability for the mortality analysis in question. In Section 4 the different fittings are compared by means of the usual non-parametric tests, and in Section 5 the most relevant conclusions are presented.

2 A review of parametric models for mortality graduation

The representation of mortality data by means of parametric models attracted the attention of actuaries, demographers and statisticians throughout the past century. These methods are based on the hypothesis that the chosen measurement of mortality is a function of age, x, $f_\alpha(x)$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ being parameters to be determined. In short, obtaining the graduation consists of applying the regression techniques which are widely described and used in statistics literature to the particular case of mortality data.

The objective of applying these procedures is to obtain the best possible fitting with the minimum number of parameters. It is therefore necessary to obtain a balance between the number of parameters and the goodness of fit. Congdon (1993) warns how many demographic graduation studies have emphasized the goodness of fit without considering the statistical stability of the parameters involved in the regression, usually leading to an overparameterization of the model, which shows up when the following are observed: standard errors for the parameter estimates that are too big, high correlations between the parameter estimates and failures of convergence in the iterative routines of non-linear fitting. An overparameterization also has practical implications on the use of graduation. For example, in the comparison of the time series of the parameters obtained when fitting mortality data corresponding to different years, the prediction of values for future years can show irregular erratic fluctuations that can make prediction difficult. There is also a strong relation between overparameterization and the instability of the parameters over time. There are therefore reasons to prefer parsimonious functions, with few parameters, despite producing slight losses in the goodness of fit.

The form of the functions that fit the data are diverse and fundamentally based on the profile presented by the crude estimations of the mortality measure used. The different models proposed by various authors are collected together in Gerber (1997) and Benjamin and Pollard (1992).

2.1 CMI Bureau Methodology

The Continuous Mortality Investigation (CMI) Bureau of the Institute and Faculty of Actuaries of London was created in 1924, when the continuous collection of mortality data began. It is responsible for constructing standard life tables for use in Great Britain's insurance industry. Forfar *et al.* (1988) have given an easily-understood description of the methodology that is normally used by the CMI to produce such tables. This methodology is a generalization of the Gompertz (1825) and Makeham (1860) models. It was applied to Spanish data by Navarro (1991) and to data of the Valencia Region by Navarro *et al.* (1995).

In order to get the graduation, the CMI Bureau uses the *Gompertz-Makeham functions of the type (r,s)*. They are functions with $r + s$ parameters of the form

$$GM_{\alpha}^{r,s}(x) = \sum_{i=1}^{r} \alpha_i x^{i-1} + \exp\left(\sum_{j=r+1}^{r+s} \alpha_i x^{j-r-1}\right),$$

with the convention that if $r = 0$, they only present an exponential part, and if $s = 0$, they only possess a polynomial term. The *Logit Gompertz-Makeham of the type (r,s)* are alternative models that can be derived from the GM functions, the general expression of which is

$$LGM_{\alpha}^{r,s}(x) = \frac{GM_{\alpha}^{r,s}(x)}{1 + GM_{\alpha}^{r,s}(x)}.$$

In order to estimate the value of the parameters included in these functions, two optimization criteria are considered, that of maximum likelihood or that of minimum χ^2, which in practice produce very similar graduations (a detailed discussion is presented by Forfar *et al.*, (1988)). The minimum χ^2 criterium is the usual χ^2 statistic, that is the sum of squared standardized residuals.

This methodology can be reformulated and extended by using the schemes of generalized linear and non-linear models. The experience in graduation using generalized linear models has been compiled in actuarial literature by Renshaw (1991), Renshaw and Hatzopoulos (1996), Renshaw *et al.* (1997) and Verrall (1996). The use of generalized linear models (GLM) for the graduation of both the probability of death at age x, q_x, and the force of mortality at age x, μ_x, is justified because both response variables are not normal. Details about modelling and probability distribution assumptions for both mortality measures follow.

2.1.1 GLM for μ_x

Let us suppose that E_x^c persons enter observation under hypothesis that the force of mortality (instantaneous mortality rate) is a constant, $\mu_{x+\frac{1}{2}}$, during the period of observation and that the death or survival of each one is independent. In this case E_x^c represents those central exposed to risk, which can get modified throughout the duration of the study, meaning that the number of individuals in the study is not determined. The number of deaths which occur in the period of observation, D_x, will have a Poisson distribution with average and variance equal to $E_x^c\mu_{x+\frac{1}{2}}$. We consider the graduation of μ_x, with $D_x \sim Po(E_x^c\mu_{x+\frac{1}{2}})$ independent, the link utilized being $log\left(\mu_{x+\frac{1}{2}}\right)$, which is the canonical link of the Poisson family, and the model which is used is $\mu_{x+\frac{1}{2}} = GM(r, s)$, which gives rise to a linear predictor when $r = 0$.

When the predictor is not linear, Renshaw (1991) suggests an iterative method which enables the application of a similar methodology that is based on Makeham's historical formula $\eta_x = A + Bc^x$. Given that is not possible to transform this non-linear form into a linear one unless $A = 0$, it is possible to introduce a trivial reparametrization in exponential form and write

$$\eta_x = \alpha + \beta exp(\phi x). \tag{1}$$

The non-linear term

$$g(x;\phi) = exp(\phi x), \tag{2}$$

can be approximated

$$g(x;\phi) \simeq g(x;\phi_0) + (\phi - \phi_0)\left(\frac{\partial g}{\partial \phi}\right)_{\phi=\phi_0},$$

so that $\beta exp(\phi x)$ can be replaced in (1) by $\beta u + \gamma v$ with

$$u = g(x;\phi_0), \qquad v = \left(\frac{\partial g}{\partial \phi}\right)_{\phi=\phi_0} \qquad \text{and} \qquad \gamma = \beta(\phi - \phi_0)$$

In this way, the non-linear term (2) has been converted into a linear expression which can be inserted in the predictor of a generalized linear model.

So, starting from an initial value ϕ_0, we calculate the covariables

$$u = \exp(\phi_0 x) \quad \text{and} \quad v = x\exp(\phi_0 x),$$

and the parameters β and γ estimated following the adjustment of the model as in any linear estimator.

We then update

$$\phi_1 = \phi_0 + \frac{\hat{\gamma}}{\hat{\beta}}$$

and this process is repeated until convergence, which is not guaranteed for very distant initial values. We found that an initial value of $\phi_0 = 0.0005$ produced convergence in many of the sets of typical data which we graduated in this way. This method enables the graduation of μ_x, with a Poisson distribution and identity link, through models $GM_x(r, 2)$ with $r \neq 0$.

Another alternative consists of considering D_x as fixed and equal to the number of observed deaths, d_x, and assuming therefore that E_x^c follows a Gamma distribution with parameters $\alpha = d_x$ y $\beta = \mu_{x+\frac{1}{2}}$. Gerber (1997) considered this distribution and it was used by Renshaw *et al.* (1997) to graduate $1/\mu_x$, force of vitality according to Lambert (1772) terminology, through a generalized linear model. We can therefore use response E_x^c variables with averages $\lambda_x = d_x \dfrac{1}{\mu_{x+\frac{1}{2}}}$, variances $\sigma_x^2 = d_x \dfrac{1}{\mu^2_{x+\frac{1}{2}}}$ and weights $\omega_x = d_x$. Taking the log link, we get

$$\log \lambda_x = \log d_x - \log \mu_{x+\frac{1}{2}} = \log d_x + \eta_x,$$

where η_x is the linear predictor. Renshaw (1991) obtained results for his data which were very similar to both of the μ_x graduation proposals.

2.1.2 GLM for q_x

Let us suppose that E_x^i persons come under observation at age x and continue under observation until they survive to $x + 1$ or die before. In this case we denote initial exposed to risk as E_x^i, which determines the number of individuals in the study. Also, let us suppose that the probability of death during the year for each one of them is q_x, and that the death or survival of one is independent of the death or survival of the others. If we call D_x the random variable which represents the number of deaths that occur in the year, we will get $D_x \sim B(E_x^i, q_x)$.

We perform the graduation of q_x using the function

$$q_x = LGM(r, s) = \frac{GM(r, s)}{1 + GM(r, s)}, \tag{3}$$

using the transformation logit(q_x) as the link, which is the canonical link of the binomial family. From (3) we easily obtain

$$\frac{q_x}{1 - q_x} = GM(r, s),$$

so that if $r = 0$, logit(q_x) corresponds to a linear predictor.

Heligman and Pollard's laws. An alternative to the previous functions are the Heligman and Pollard laws (Heligman and Pollard, 1980). These laws have been used by various countries (England, Sweden, Germany, Spain, United States of America and Australia) since the UN promoted the fitting of mortality through Heligman and Pollard's first law. Heligman and Pollard, inspired by Thiele (1972), adjusted a new mortality law to post-war Australian data with the general expression

$$\frac{q_x}{1 - q_x} = \sum_{i=1}^{n} A_i \exp\left(-B_i \left(f_i(x) - C_i\right)^{D_i}\right),$$

where A_i, B_i, C_i, D_i, $i = 1, 2, ..., n$, are the parameters to be estimated, and where $f_i(x)$ is usually the identity function, $f_i(x) = x$, or $f_i(x) = \ln(x)$

The three expressions that really fitted Australian mortality were as follows:

Heligman and Pollard's first law

$$\frac{q_x}{1-q_x} = A^{(x+B)^C} + D\exp\left(-E(\ln x - \ln F)^2\right) + GH^{(x-x_0)}$$

an expression that they consider cannot be distinguished from

$$q_x = A^{(x+B)^C} + D\exp\left(-E(\ln x - \ln F)^2\right) + \frac{GH^x}{1+GH^x}.$$

Heligman and Pollard's second law

$$q_x = A^{(x+B)^C} + D\exp\left(-E(\ln x - \ln F)^2\right) + \frac{GH^x}{1+KGH^x} \tag{4}$$

Heligman and Pollard's third law

$$q_x = A^{(x+B)^C} + D\exp\left(-E(\ln x - \ln F)^2\right) + \frac{GH^{x^k}}{1+GH^{x^k}}$$

The first term models childhood mortality, the second one the accident hump and the third term natural mortality caused by senescence (Heligman and Pollard, 1980). The graph in Figure 1 shows this decomposition. The interpretation of the parameters is as follows: A represents the infant mortality rate; B represents death probability for children who are 1 year old; C is closely related with the rate at which an individual adapts to his environment, three parameters taking values in the interval (0,1). D, E and F are referred as the accident hump, D indicates the severity of the accident hump with values in (0,1), E with large values, in $(0,\infty)$, indicate a concentrated accident hump and F from 15 to advanced age indicates the location of the hump maximum. Finally, G indicates the base level of later adult mortality, and H is the rate of increase in mortality at the later adult ages and its domains are (0,1) and $(0,\infty)$ respectively.

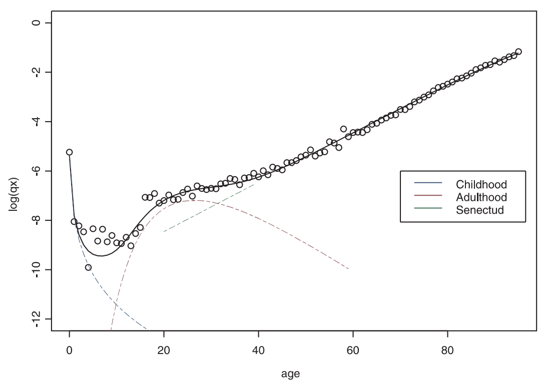

***Figure 1**: Decomposition of Heligman and Pollard's Law.*

In order to estimate the coefficients, given the heterocedasticity of the data, the error structure should be accommodated by differential weighting of the rates for different ages (Congdon, 1993). Using criteria of weighted least squares $WLS(\alpha) = \sum_x \omega_x(\dot{q}_x - f_\alpha(x))^2$, with $f_\alpha(x)$ as in Heligman and Pollard's laws, and weights inversely proportional to binomial sampling variances

$$var(\dot{q}_x) = \frac{\dot{q}_x(1-\dot{q}_x)}{e_x},$$

and taking into consideration that $(1 - q_x) \approx 1$, we obtain the following expressions:

$$\begin{aligned} a) &\quad \sum_x \frac{e_x}{\dot{q}_x}(\dot{q}_x - f_\alpha(x))^2 \\ b) &\quad \sum_x (\dot{q}_x - f_\alpha(x))^2 \\ c) &\quad \sum_x \frac{e_x}{\dot{q}_x^2}(\dot{q}_x - f_\alpha(x))^2 \\ d) &\quad \sum_x \frac{1}{\dot{q}_x^2}(\dot{q}_x - f_\alpha(x))^2 \end{aligned} \qquad (5)$$

including unweighted least squares in item *b)*. In these expressions, $\dot{q}_x$ is the crude estimate of q_x and e_x is the estimate of initial exposure to risk E_x^i.

An example of the application of these laws to mortality data of our geographic and social surroundings can be found in Felipe and Guillén (1999). They apply the second law to Spanish data for the period 1979-82. A Bayesian approach for Heligman and Pollard's laws has been proposed by Dellaportas *et al.* (2001), using Markov chain Monte Carlo simulation for avoiding the numerical problems that arise in classical methods.

3 Application to mortality data of the Valencia Region

The comparative study of the different parametric models of graduation is done by applying them to the mortality data of the Valencia Region, using aggregate population and death figures corresponding to the three-year period 1999-2001. These two data sets were published by the Spanish National Institute of Statistics (INE) and are classified by age (ranging from 0 to 100 or older) and sex. They both refer to the Valencia Region as the place of residence, which means the two sets of figures correspond to each other coherently.

As the population census takes place every 10 years and during the first year of the ten year period, only the data for 2001 are real counts, the data for 1999 and 2000 being

Table 1: *Age, x, initial number exposed to risk, e_x, and number of deaths, d_x, observed in the period 1999-2001.*

	MEN		WOMEN			MEN		WOMEN	
x	e_x	d_x	e_x	d_x	Age	e_x	d_x	e_x	d_x
0	39199.70	180.00	36955.70	182.00					
1	38315.50	21.00	36211.50	17.00	49	48797.50	212.00	50280.50	137.00
2	38139.50	5.00	35822.50	11.00	50	48223.00	234.00	49786.00	132.00
3	38096.00	7.00	35797.00	10.00	51	47697.50	254.00	49363.50	151.00
4	38345.00	6.00	36114.50	6.00	52	47301.50	298.00	49073.50	160.00
5	39066.00	8.00	36755.50	7.00	53	46565.50	265.00	48400.50	180.00
6	39971.50	9.00	37689.00	1.00	54	45335.00	318.00	47270.00	195.00
7	40772.50	10.00	38576.00	11.00	55	43837.00	345.00	45854.00	189.00
8	41449.00	8.00	39300.50	7.00	56	42854.00	349.00	44903.00	205.00
9	42046.50	2.00	39956.00	8.00	57	42080.50	376.00	44140.00	199.00
10	42696.50	8.00	40522.50	9.00	58	40376.50	359.00	42484.00	227.00
11	43574.50	5.00	41207.00	3.00	59	38943.50	457.00	41196.00	268.00
12	44781.00	5.00	42271.00	4.00	60	38785.00	440.00	41292.50	248.00
13	46223.00	10.00	43662.50	9.00	61	38629.50	476.00	41424.50	303.00
14	47927.00	11.00	45348.50	8.00	62	38341.50	487.00	41368.00	347.00
15	50014.50	20.00	47414.00	19.00	63	39109.00	591.00	42455.00	398.00
16	52505.50	34.00	49799.50	19.00	64	39885.50	656.00	43643.50	416.00
17	55265.50	50.00	52413.50	26.00	65	39758.50	706.00	43911.00	488.00
18	58202.00	66.00	55256.00	24.00	66	39325.00	744.00	43958.00	507.00
19	61265.00	49.00	58248.50	29.00	67	38834.50	868.00	43998.50	614.00
20	64133.00	66.00	61040.00	30.00	68	37958.00	939.00	43610.00	665.00
21	66470.00	60.00	63360.50	22.00	69	36676.50	922.00	42844.00	748.00
22	68160.50	58.00	65063.50	36.00	70	35290.50	994.00	42032.50	821.00
23	68999.50	59.00	66045.50	31.00	71	33869.00	1043.00	41144.50	881.00
24	69101.50	70.00	66303.00	31.00	72	32269.50	1146.00	40023.00	977.00
25	68737.50	56.00	66035.00	33.00	73	30646.50	1245.00	38763.00	1136.00
26	68178.00	71.00	65598.00	29.00	74	29109.00	1276.00	37516.50	1281.00
27	67544.00	73.00	65109.00	33.00	75	27468.00	1273.00	36204.50	1383.00
28	66988.50	69.00	64620.50	40.00	76	25565.00	1327.00	34564.00	1542.00
29	66568.50	81.00	64415.50	40.00	77	23332.00	1484.00	32491.50	1579.00
30	66311.00	71.00	64476.50	37.00	78	20987.50	1488.00	30287.50	1715.00
31	66111.00	99.00	64526.00	41.00	79	18440.50	1290.00	27841.50	1704.00
32	65994.00	77.00	64630.00	52.00	80	15972.00	1199.00	25308.50	1796.00
33	65963.00	93.00	64775.00	65.00	81	13771.50	1148.00	22870.00	1856.00
34	65564.00	111.00	64497.50	41.00	82	12148.50	1169.00	20833.00	2095.00
35	64618.50	109.00	63797.00	75.00	83	10820.00	1040.00	18958.50	2096.00
36	63513.00	123.00	63059.50	72.00	84	9785.50	1094.00	17317.00	2256.00
37	62503.50	102.00	62347.00	68.00	85	8764.50	1074.00	15721.50	2325.00
38	61467.00	117.00	61572.50	69.00	86	7716.50	1021.00	14113.00	2332.00
39	60471.00	128.00	60830.50	83.00	87	6670.50	973.00	12400.50	2306.00
40	59447.50	136.00	59983.00	89.00	88	5589.50	845.00	10596.00	2141.00
41	58063.00	128.00	58719.00	85.00	89	4631.00	733.00	8933.00	2051.00
42	56272.50	164.00	57007.00	90.00	90	3737.00	593.00	7320.00	1809.00
43	54271.00	162.00	55168.50	104.00	91	2995.50	550.00	5883.00	1654.00
44	52509.00	165.00	53575.00	102.00	92	2301.00	462.00	4558.50	1396.00
45	51236.50	165.00	52366.50	122.00	93	1702.50	334.00	3416.50	1097.00
46	50214.50	173.00	51369.50	118.00	94	1269.50	234.00	2535.00	936.00
47	49358.50	202.00	50587.50	116.00	95	852.50	179.00	1740.00	667.00
48	48993.50	188.00	50376.00	133.00	96	576.50	130.00	1148.50	530.00

	MEN										
Poisson (GLM)		**GM(0,2)**	**GM(0,3)**	**GM(0,4)**	**GM(0,5)**	**GM(0,6)**	**GM(0,7)**	**GM(0,8)**	**GM(0,9)**	**GM(0,10)**	**GM(0,11)**	**GM(0,12)**
deviance	102212	2144.98	960.44	793.61	782.45	636.61	519.51	379.36	227.25	174.14	165.85	165.35
d.f.	96	95	94	93	92	91	90	89	88	87	86	85
log-likelihood	218846.9	217774.4	218366.7	218450.1	218455.7	218528.6	218587.1	218657.2	218733.3	218759.8	218764	218764.2
χ^2		13552.89	2348.77	1202.75	1058.72	685.13	530.36	380.66	224.14	176.54	170.07	169.22
Poisson (GNLM)			**GM(1,2)**		**GM(3,2)**	**GM(4,2)**	**GM(5,2)**	**GM(6,2)**	**GM(7,2)**	**GM(8,2)**	**GM(9,2)**	**GM(10,2)**
deviance	102212		862.7652		736.7106	605.2137	342.1944	311.1831	339.0331	330.6377	318.0727	274.3146
d.f.	96		95		94	93	92	91	90	89	88	87
log-likelihood	218846.9		218415.5		21478.5	218544.3	218675.8	218691.3	218677.4	218681.6	218687.9	218709.7
χ^2			1378.717		1066.584	803.3898	347.6592	280.7295	338.2825	324.1219	309.911	268.9084
Gamma (GLM)		**GM(0,2)**	**GM(0,3)**	**GM(0,4)**	**GM(0,5)**	**GM(0,6)**	**GM(0,7)**	**GM(0,8)**	**GM(0,9)**	**GM(0,10)**	**GM(0,11)**	**GM(0,12)**
deviance	102212	2311.18	1426.80	1423.07	1547.64	1392.25	1156.79	731.03	323.09	271.37	279.22	278.12
d.f.	96	95	94	93	92	91	90	89	88	87	86	85
log-likelihood	218846.9	217691.3	218133.5	218135.4	218073.1	218150.8	218268.5	218481.4	218685.4	218711.2	218707.3	218707.8
χ^2		10391.08	7198.96	7846.74	6854.49	3499.79	1885.87	933.81	381.43	318.12	331.78	331.58
	WOMEN											
Poisson (GLM)		**GM(0,2)**	**GM(0,3)**	**GM(0,4)**	**GM(0,5)**	**GM(0,6)**	**GM(0,7)**	**GM(0,8)**	**GM(0,9)**	**GM(0,10)**	**GM(0,11)**	**GM(0,12)**
deviance	172821.4	4481.88	1133.22	863.537	862.63	409.18	324.47	283.97	185.99	155.38	113.46	113.46
d.f.	96	95	94	93	92	91	90	89	88	87	86	85
log-likelihood	312737.5	310496.6	312170.9	312305.8	312306.2	312532.9	312575.3	312595.6	312644.5	312659.8	312680.8	312680.8
χ^2		100664.9	2573.70	1151.41	1198.49	414.03	317.73	276.38	178.98	152.13	111.54	111.53
Poisson (GNLM)			**GM(1,2)**		**GM(3,2)**	**GM(4,2)**	**GM(5,2)**	**GM(6,2)**	**GM(7,2)**	**GM(8,2)**	**GM(9,2)**	**GM(10,2)**
deviance	172821.4		1129.71		736.71	611.27	603.07	352.42	337.97	270.23	262.90	215.51
d.f.	96		95		94	93	92	91	90	89	88	87
log-likelihood	312737.5		312172.7		310226.6	312431.9	312436	312561.3	312568.6	312602.4	312606.1	312629.8
χ^2			1782.39		1066.58	787.88	771.41	390.49	362.17	274.40	261.50	206.54
Gamma (GLM)		**GM(0,2)**	**GM(0,3)**	**GM(0,4)**	**GM(0,5)**	**GM(0,6)**	**GM(0,7)**	**GM(0,8)**	**GM(0,9)**	**GM(0,10)**	**GM(0,11)**	**GM(0,12)**
deviance	172821.4	11628.81	1694.593	1691.821	2014.139	1080.164	1018.142	897.1084	408.2796	203.1785	173.3137	159.9273
d.f.	96	95	94	93	92	91	90	89	88	87	86	85
log-likelihood	312737.5	306923.1	311890.2	311891.6	311730.5	312197.5	312228.5	312289	312533.4	312635.9	312650.9	312657.6
χ^2		29977.59	7965.66	9101.77	8058.54	2837.94	1600.56	1048.8	438.68	220.34	188.99	180.21

Table 2: *Goodness of fit measurements for the different models of* μ_x.

inter-census estimations obtained from various INE publications, (INE, 1997) and (INE, 2001).

The first step is to calculate the crude estimates of q_x from these data. From among the different existing proposals for carrying out such estimates, we have used that of Navarro *et al.* (1995):

$$\dot{q}_x = \frac{D_{x(t-1)} + D_{xt}}{1/2P_{x(t-1)} + P_{xt} + 1/2P_{x(t+1)} + 1/2(D_{x(t-1)} + D_{xt})}, \tag{6}$$

where P_{xt} is the population of people whose ages are between x and $x + 1$ years old on 1st January of the year t, and D_{xt} is the number of individuals deaths whose ages were between x and $x + 1$ during the year t. This choice is made because as we do not have the deaths classified according to the year of birth, but according to age and sex, the expression (6) allows us to avoid this difficulty because it supposes uniform death distribution throughout the year. The denominator of the expression is e_x, an estimation of E^i_x.

The same expression, adequately corrected in denominator, can be used for the crude estimation of μ_x,

$$\dot{\mu}_x = \frac{D_{x(t-1)} + D_{xt}}{1/2P_{x(t-1)} + P_{xt} + 1/2P_{x(t+1)}}, \tag{7}$$

where the denominator is now $e_x - d_x/2$, an estimation of E^c_x.

The graphic representation of the logarithms of the crude estimations led us to take a range of between 0 to 96 years old for age, which seems to us compatible with the use of the maximum possible and with the demand for relatively stable behavior. Beyond this age the logarithms decrease, showing behaviour which is difficult to explain.

In the period under study, there were nearly 3.96 million men and 4.11 million women exposed to risk. 77% of these were over 20 years of age. In the same period, nearly 39.3 thousand men and 51.4 women died, with the great majority, approximately 99%, doing so after the age of 20 (see Table 1).

3.1 Modelling μ_x

The modelling μ_x has been done by means of $GM(r, s)$ functions, using GLM and generalized non-linear models (GNLM) of the Poisson and Gamma families. The goodness of fit of the models involved has been measured by means of the log-likelihood and the χ^2. Since the fitting must improve as the number of parameters increase, we must to see if that improvement is significant and to do so we use the deviance and Mallow's C_p statistic, both testing the improvement of the fitting in relation to the increase in complexity of the model.

Table 2 summarizes the results obtained. It is divided into two parts, according to sex, and then each part into three groups of results.

1. Those corresponding to the functions $GM(0, s)$, $s = 2, \ldots, 12$, fitted through the generalized linear models of the Poisson family using the log as a link.
2. Those corresponding to the functions $GM(r, 2)$, $r = 1, \ldots, 10$, fitted through generalized non linear models of the Poisson family using the identity as a link;
3. Those corresponding to the functions $GM(0, s)$, $s = 2, \ldots, 12$ fitted through generalized linear models of the Gamma family using the log as a link, even though what we adjust in this case is the vitality force, $1/\mu_x$.

The first column of the table contains the initial reference values corresponding to the null model for the deviance and to the saturated model for the log-likelihood. We conclude that the best model is $GM(0, 11)$, obtained through generalized linear models using the Poisson family ($GM(0, 12)$ has an insignificant improvement of deviance. Once its coefficients have been calculated, we test that they are significant for both sexes in Table 3.

We should point out that for making the results of Poisson and Gamma models comparable, we have evaluated the inverse of the Gamma model predictions. Thus, the results shown in Table 2 have been calculated from the Poisson Likelihood obtained with these inverses.

Figure 2 shows the graphic comparison of the $GM(0, s)$ models, from $s = 7$ to $s = 12$ for each sex. In order to make the results obtained with all models and functions comparable, the above comparison is made in terms of q_x in place of the fitted μ_x.

Table 3: *Coefficients of models* $GM(0, 11)$

	MEN[a]				WOMEN[b]			
	coef	**std error**	**p-value**	**t-value**	**coef**	**std error**	**t-value**	**p-value**
const.	-5.439e+00	1.046e-01	-51.985	< 2e-16	-5.369e+00	8.434e-02	-63.655	< 2e-16
age	-1.874e+00	1.613e-01	-11.614	< 2e-16	-1.902e+00	1.313e-01	-14.481	< 2e-16
age^2	3.165e-01	3.776e-02	8.383	8.85e-13	3.430e-01	3.137e-02	10.935	< 2e-16
age^3	-2.428e-02	3.788e-03	-6.409	7.55e-09	-2.917e-02	3.174e-03	-9.192	2.00e-14
age^4	1.061e-03	2.081e-04	5.098	2.02e-06	1.418e-03	1.742e-04	8.139	2.77e-12
age^5	-2.883e-05	6.910e-06	-4.172	7.20e-05	-4.257e-05	5.743e-06	-7.412	8.04e-11
age^6	5.060e-07	1.448e-07	3.494	0.000755	8.187e-07	1.191e-07	6.875	9.34e-10
age^7	-5.756e-09	1.929e-09	-2.984	0.003703	-1.012e-08	1.566e-09	-6.461	5.99e-09
age^8	4.106e-11	1.584e-11	2.592	0.011210	7.776e-11	1.268e-11	6.134	2.54e-08
age^9	-1.672e-13	7.316e-14	-2.285	0.024762	-3.386e-13	5.767e-14	-5.871	7.94e-08
age^{10}	2.968e-16	1.454e-16	2.041	0.044275	6.384e-16	1.128e-16	5.657	1.98e-07

a. deviance= 165.85 on 86 d. f.; over-dispersion parameter $\phi = 1.977649$

b. deviance= 113.46 on 86 df; over-dispersion parameter $\phi = 1.296974$

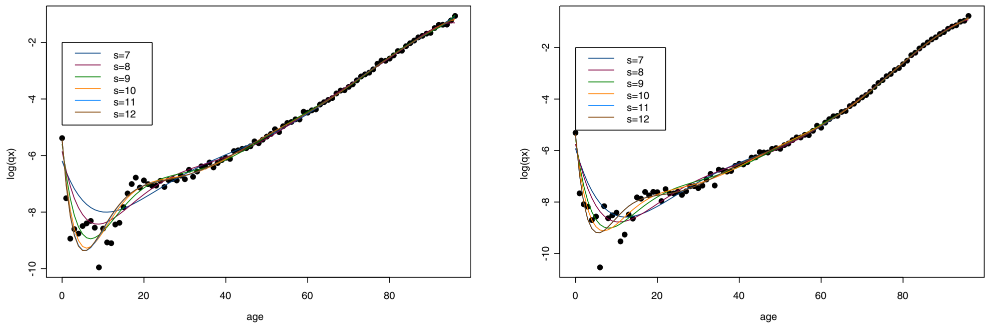

Figure 2: *Comparison of the* q_x *corresponding to the models GM(0,s) for men and women.*

3.2 Modelling q_x

The modelling q_x has been done through the functions $LGM(0, s)$, $s = 2, \ldots, 12$, using generalized linear models of the binomial family, and through Heligman and Pollard's second law for whose estimation we have used weighted least squares. Table 4 summarizes the results obtained. The first column of the table contains the initial reference values corresponding to the null model for the deviance and to the saturated model for the log-likelihood. The observed values indicate that the best model for both sexes, taking into consideration the commitment between goodness of fit and its complexity, is $LGM(0, 11)$. The coefficients of these models for both sexes are shown in Table 5.

Figure 3 shows the graphic comparison of the models from $s = 7$ to $s = 12$ for each sex. Both are presented in *logit* scale.

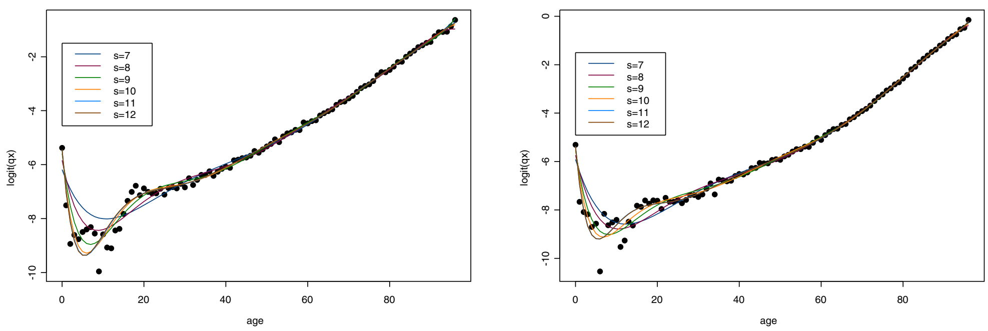

Figure 3: *Comparison of models LGM (0,s) for men and women*

	MEN				WOMEN			
		LGM(0,2)	**LGM(0,3)**	**LGM(0,4)**		**LGM(0,2)**	**LGM(0,3)**	**LGM(0,4)**
deviance	102248	2332.71	890.30	794.34	172974.9	5080.83	886.03	806.22
d.f.	96	95	94	93	96	95	94	93
log-likelihood	-169352.1	-170518.5	-16797.3	-16749.3	-190038.2	-192578.6	-190481.2	-190441.3
χ^2		14214.66	2042.38	1226.10		121027.5	1839.7	1195.73
	LGM(0,5)	**LGM(0,6)**	**LGM(0,7)**	**LGM(0,8)**	**LGM(0,5)**	**LGM(0,6)**	**LGM(0,7)**	**LGM(0,8)**
deviance	769.87	635.95	517.09	375.22	792.99	408.35	323.94	279.36
d.f.	92	91	90	89	92	91	90	89
log-likelihood	-169737.1	-169670.1	-169610.7	-169539.7	-190434.7	-190242.3	-190200.1	-190177.8
χ^2	1000.90	683.45	527.72	371.51	1047.88	411.94	317.108	268.03
	LGM(0,9)	**LGM(0,10)**	**LGM(0,11)**	**LGM(0,12)**	**LGM(0,9)**	**LGM(0,10)**	**LGM(0,11)**	**LGM(0,12)**
deviance	225.16	173.95	166.14	165.59	186.84	155.37	114.16	114.10
d.f.	88	87	86	85	88	87	86	85
log-likelihood	-169464.7	-169439.1	-169435.2	-169434.9	-190131.6	-190115.8	-190095.2	-190095.2
χ^2	221.32	175.54	169.22	168.31	179.15	151.64	111.77	111.82

Table 4: *Goodness of fit measures for the* $LGM(0, s)$ *of* q_x

Table 5: *Coefficients of models* $LGM(0, 11)$

	MEN[a]				WOMEN[b]			
	coef	**std error**	**t-value**	**p-value**	**coef**	**std error**	**t-value**	**p-value**
const.	-5.438e+00	1.049e-01	-51.825	< 2e-16	-5.367e+00	8.476e-02	-63.316	< 2e-16
age	-1.875e+00	1.623e-01	-11.549	< 2e-16	-1.913e+00	1.328e-01	-14.401	< 2e-16
age^2	3.167e-01	3.814e-02	8.305	1.27e-12	3.467e-01	3.190e-02	10.868	< 2e-16
age^3	-2.430e-02	3.842e-03	-6.326	1.09e-08	-2.962e-02	3.244e-03	-9.132	2.65e-14
age^4	1.063e-03	2.119e-04	5.015	2.82e-06	1.446e-03	1.789e-04	8.081	3.63e-12
age^5	-2.890e-05	7.063e-06	-4.092	9.63e-05	-4.360e-05	5.928e-06	-7.355	1.05e-10
age^6	5.078e-07	1.486e-07	3.417	0.000968	8.421e-07	1.235e-07	6.817	1.21e-09
age^7	-5.783e-09	1.986e-09	-2.911	0.004583	-1.045e-08	1.632e-09	-6.402	7.79e-09
age^8	4.131e-11	1.637e-11	2.523	0.013468	8.059e-11	1.327e-11	6.072	3.32e-08
age^9	-1.684e-13	7.586e-14	-2.220	0.029045	-3.522e-13	6.066e-14	-5.806	1.05e-07
age^{10}	2.994e-16	1.513e-16	1.980	0.050947	6.662e-16	1.192e-16	5.588	2.66e-07

a. deviance= 166.14 on 86 d. f.; over-dispersion parameter $\phi = 1.980545$

b. deviance= 114.16 on 86 d. f.; over-dispersion parameter $\phi = 1.304967$

The graduation results for Heligman and Pollard's second law are presented graphically in Figure 4. The criterion used for weighting the square difference was the first in 5), the choice being based on the number of relative deviations greater than 2 and 3 and the value of the χ^2 for the goodness of fit.

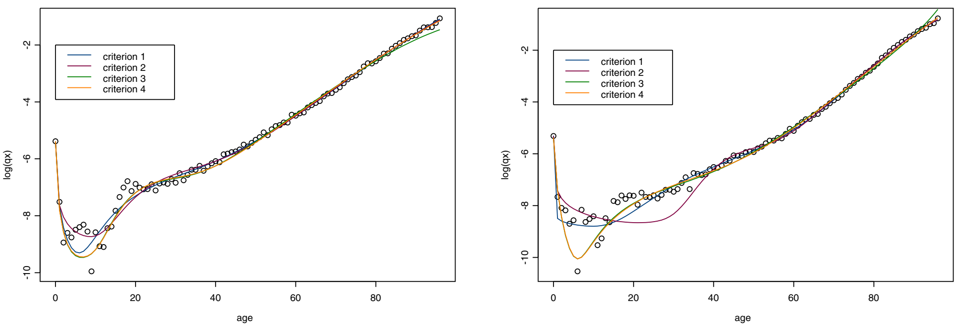

Figure 4: *Comparison of Heligman and Pollard's models for men and women*

The coefficients corresponding to the Heligman-Pollard model have not presented great difficulty for men. This was not the case for women because they do not present the accident hump. The Spanish female population has high mortality spread over many more years (Felipe and Guillén, 1999). The problem was solved by fixing the parameter $F = 96$. This technique of fixing the values of some parameters and fitting the rest was used by Congdon (1993). In order not to fall into the problem pointed out by Congdon (1993), we have also carried out a study of the meaningfulness of the parameters. Some problems related to the singularity of the matrix of the coefficients were found in this study, and have been overcome through the use of generalized non-linear least squares. The parameter estimates are shown in Table 6, some of them not significant, in particular, A, B, C, D and E for men, and B and C for women.

Table 6: *Coefficients of Heligman and Pollard models*

	MEN				WOMEN			
	coef	**std error**	**t-value**	**p-value**	**coef**	**std error**	**t-value**	**p-value**
A	0.00054	0.00035	1.5670	0.1207	0.000335	0.0000746	4.4886	<0.0001
B	0.12921	0.21037	0.6142	0.5407	0	0.0000030	0.1073	0.9148
C	0.16301	0.09059	1.7993	0.0754	0.027444	0.0153661	1.7860	0.0775
D	0.00138	0.00065	2.1289	0.0361	0.002757	0.0003135	8.7939	<0.0001
E	0.74764	0.44753	1.6706	0.0984	1.140099	0.1308293	8.7144	<0.0001
F	63.03293	37.14621	1.6969	0.0933	96	—	—	—
G	0.00002	0	3.4307	0.0009	0.000001	0.0000001	4.5996	<.0001
H	1.11313	0.00425	262.0285	< .0001	1.159430	0.0031811	364.4753	<.0001
K	0.91755	0.26233	3.4977	0.0007	1.108379	0.0822712	13.4723	<.0001

4 Comparison of the models

We have compared the different models by choosing the best fitting model for each one of them. Specifically, we have compared the $GM(0, 11)$ for μ_x, the $LGM(0, 11)$ for q_x and Heligman and Pollard's second law (HP) for q_x. In order to make the first one comparable to the other two, the values of μ_x have been transformed through the relation

$$q_x = 1 - \exp(-\mu_{x+\frac{1}{2}}).$$

The comparison is carried out by applying the tests proposed by Forfar *et al.* (1988), which Navarro (1991) and Navarro *et al.* (1995) also used in their work. In order to obtain an expected number of deaths not inferior to 5, we have had to aggregate data for ages between 4 and 10, with the consequent decrease in the number of degrees of freedom.

We have also obtained the values of the mean absolute percentage error (MAPE) and R^2 that Felipe and Guillén (1999) used in their work. The value of R^2 has been obtained as 1 minus the proportion of the variance that remains unexplained, because if we calculate it directly as a percentage of explained variance, in some cases it exceeded 1. This can happen when the models are not linear.

Table 7 presents the results of the tests for the three models. Figure 5 shows the autocorrelations of standarized residuals for all the models. In all the cases there are a few isolated correlated values out of the Heligman and Pollard model adjusted for women. This agrees with the worst behaviour of this adjustment.

Table 7: *Comparison of the three best fitted parametric models.*

		GM(0,11) for μ_x		LGM(0,11) for q_x		HP for q_x	
		Men	Women	Men	Women	Men	Women
Relative Desv. [a]	> 2	8	3	8	4	6	10
	> 3	3	0	3	0	6	2
Signs test	pos.(neg.)	46 (48)	53 (42)	46 (48)	53 (42)	50 (44)	47 (50)
	p-value	0.4589	0.8909	0.4589	0.8909	0.7647	0.4196
Runs test	runs	44	50	44	50	41	35
	p-value	0.4319	0.5372	0.4319	0.5372	0.3839	0.2731
K-S test[b]	K-S	0.0433	0.0316	0.0426	0.0316	0.0532	0.0825
	p-value	1	1	0.9994	1	1	0.8987
χ^2 **test** [c]	χ^2	164.32	102.44	164.18	101.07	224.31	165.56
	d.f.	83	84	83	84	85	88
	p(χ^2)	2.69e-07	0.0836	2.80e-07	0.0989	1.66e-04	1.11e-06
$\mathbf{R^2}$		0.9972	0.9991	0.9972	0.9991	0.9967	0.9981
MAPE		16.34	16.44	16.35	16.44	15.13	21.17

a. standarized residuals

b. Kolmogorov-Smirnov test

c. χ^2 statistic, sum of squared standarized residuals

5 Conclusions

From Table 7 and Figure 5, we can conclude that

1. Heligman and Pollard's models fit worse than the other two,
2. Women provide a better fitting in the three models, and
3. The model $LGM(0, 11)$ provides the most acceptable results for both sexes.

In relation to the work of other authors, we should highlight two distinctive features of the methodology presented here:

- The first one is the possibility of comparing the different models, as all of them end up producing estimates of q_x and are susceptible to having their goodness of fit measured with the same criteria.
- The second one is that all the models have been fitted for the full range of ages without the need to recur to a division into sections of that range. In this respect, it is interesting to compare our best model, the $LGM(0, 11)$, with that obtained by other authors for data of the same origin (Navarro et al, 1995). This comparison can be seen in Debón *et al.* (2003). They obtain a slightly better fit, but the resulting function presents irregularities (peaks) in the junction points between the sections due to the restrictions imposed on the functions to be fitted in each section. Moreover, the fitting of a single function entails a great saving of time.

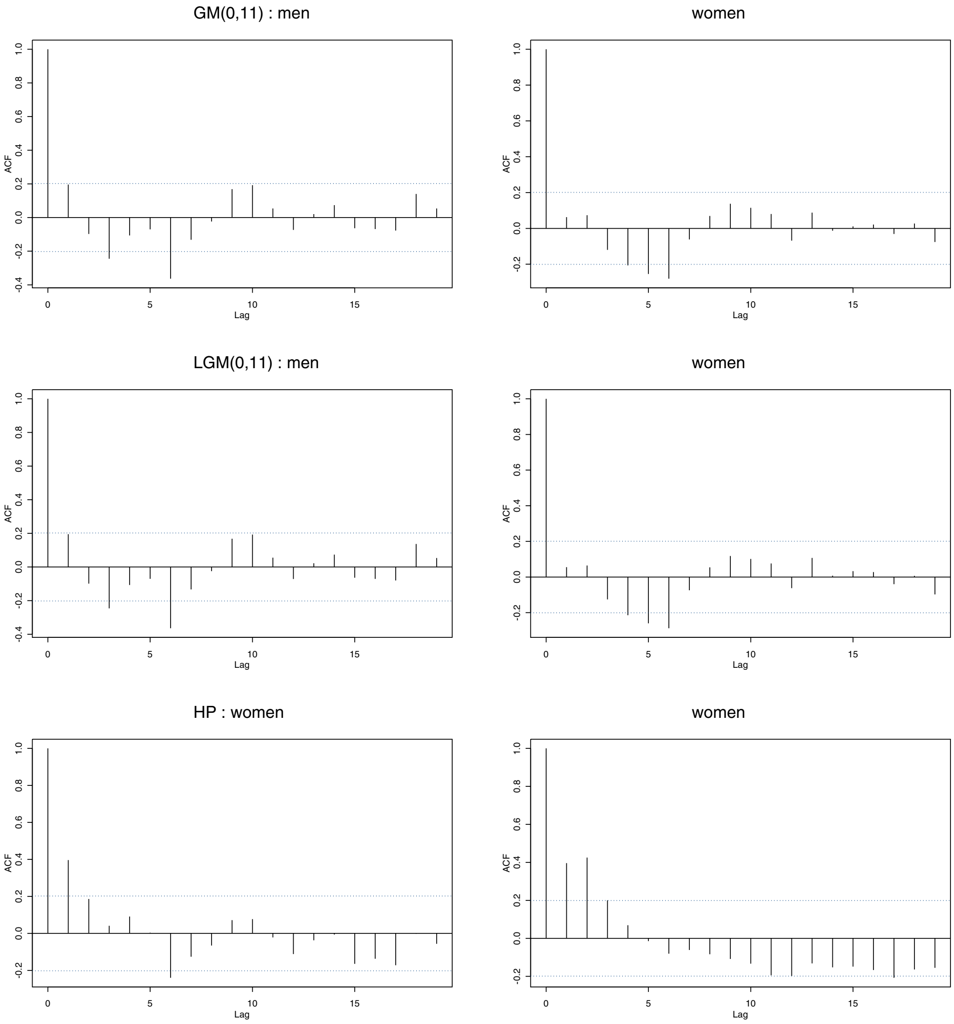

Figure 5: *Autocorrelations of standarized residuals*

We should point out that all the models present problems for younger ages due to the irregular profile of crude mortality rates. We can observe a greater distance between the values predicted by the models and the observations for the lower ages in Figures 2, 3 and 4. This is a well-known problem when graduating mortality data. Many authors achieve better fits by eliminating this group of ages, which they justify by arguing that the actuarial operations begin at a more advanced age.

Contrary to this criterion, we have decided to include the young ages groups for two reasons. The first one is that it enables as to compare our results with those obtained by Navarro *et al.* (1995), who graduate mortality data for the Valencia Region for the years 1990-92 for the complete range of ages. As far as we know, that is the only study that covers the same geographical area as ours and a comparison is vital. The second argument in favour of our approach is to remember that the double exponential which appears in Heligman and Pollard's laws was introduced specifically to deal with the difficulty of adjusting young age groups.

Finally, in line with the work of Butt and Haberman (2004), we conclude that the GLM method has a stronger theoretical justification and yields models with more favourable properties than the classical non-linear least squares method.

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