# Poverty comparisons when TIP curves intersect

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## Abstract

Non-intersection of TIP curves is recognized as a criterion to compare two income distributions in terms of poverty. The purpose of this paper it to obtain comparable poverty results for income distributions whose TIP curves intersect (possibly more than once). To deal with such situations, a sequence of higher-degree dominance criteria between TIP curves is introduced. The normative significance of these criteria is provided in terms of a sequence $C_n$ of nested classes of linear poverty measures with the property that, as the order $n$ of the class increases, the measures become more and more sensitive to the distribution of income among the poorest.

## 1. Introduction

Since the seminal paper of Sen (1976) on poverty measurement, a large body of literature dealing with this topic has been published. Because an important reason for measuring poverty is to make comparisons, part of the literature has developed by focusing on partial poverty orderings, which require unanimity in poverty rankings for a class of measures that obey some normative principles, with a fixed poverty line (see Zheng (2000) for a review of this topic).

Poverty orderings are sometimes based on comparisons of TIP curves. The TIP (Three I's of Poverty) curve (Jenkins and Lambert, 1997) cumulates the poverty gaps of the bottom $p$ proportion of the population. In order to introduce this dominance device, consider an income random variable $X$ with distribution $F$ and let $F^{-1}$ be the corresponding right continuous quantile function defined by

$$F^{-1}(t) = \sup\{x : F(x) \leq t\}, \ \ t \in [0,1].$$

Let $z > 0$ the poverty line. The proportion of poor people, $r_z(X)$, is given by

$$r_z(X) = \sup\{F(x) : x < z\},$$

and the censored quantile function $F_z^{-1}$ is defined, for all $t \in [0,1]$, as

$$F_z^{-1}(t) = \begin{cases} F^{-1}(t) & \text{if} \ \ t < r_z(F) \\ z & \text{if} \ \ t \geq r_z(F) \end{cases}.$$

Censored quantiles are, therefore, just the incomes $F^{-1}(t)$ for those in poverty (below $z$) and $z$ for those whose income exceeds the poverty line. The poverty gap associated with income $F^{-1}(t)$ is defined as $z - F_z^{-1}(t)$. The TIP curve (also sometimes referred to as the Cumulative Poverty Gap curve or the Poverty Profile curve; see Spencer and Fisher (1992), Shorrocks (1995, 1998) and Jenkins and Lambert (1998a, 1998b) associated to $X$ is given by

$$G_X(p,z) = \int_0^p \left(z - F_z^{-1}(t)\right) dt, \ \ p \in [0,1]. \tag{1}$$

In this paper, motivated by the second-order TIP dominance criterion introduced by Sordo *et al.* (2007), a family of higher-degree poverty orderings based on comparisons of TIP curves is considered. Although one finds in the literature several results concerning different notions of high-degree poverty orderings (including those by Shorrocks and Foster (1987), Foster and Shorrocks (1988) or Zheng (1999)) a higher-degree dominance criterion based on TIP curves has not been considered before. The normative significance of this family of orderings is provided in terms of a class $C$ of poverty measures which has attracted a growing interest in recent years (see Davidson and Duclos (2000), Duclos and Grégoire (2002), Duclos and Araar (2006) and Sordo *et al.* (2007)). Members of this class have the following functional form:

$$I_X(\Phi, z) = \int_0^1 \left(z - F_z^{-1}(t)\right) d\Phi(t), \tag{2}$$

where the poverty gaps are weighted with a continuous probability distribution, $\Phi$, with support $\Delta(\Phi) \subseteq [0,1]$. The class $C$, is analogous to the class of linear inequality

measures proposed by Mehran (1976) for inequality indices and Yaari (1988), for social welfare indices. Following Duclos and Araar (2006), members of $C$ satisfy the following axioms: Pareto (the measure does not increase whenever someone's income increases), focus (the measure depends only on the income of the poor), symmetry (permuting the incomes has no influence on the value of the measure) and replication invariance (the measure is not affected by the pooling of several identical populations). In addition, members of

$$C_1 = \{I(\Phi, z) \in C \text{ such that } \Phi \text{ is concave}\} \tag{3}$$

satisfy the Pigou-Dalton Principle of Transfer (any mean-preserving transfer from a poor person to a poorer person that leaves unchanged their relative rank in the distribution, must decrease poverty). Duclos and Araar (2006, Section 10.1) shows that the class $C_1$ can be characterized in terms of the first-degree TIP dominance criterion as follows:

$$G_X(p, z) \leq G_Y(p, z) \text{ for all } p \in [0, 1] \Leftrightarrow I_X(\Phi, z) \leq I_Y(\Phi, z), \text{for all } I(\Phi, z) \in C_1.$$

As shown by Sordo *et al.* (2007), when TIP curves intersect, comparable results are possible by restricting attention to a class $C_2$ whose members satisfy the Diminishing Transfer Principle, which strengthens the Pigou-Dalton Principle of Transfer by requiring that the reduction of poverty resulting from a transfer from a poor person to a poorer person is higher the poorer the recipient. Namely

$$C_2 = \{I(\Phi, z) \in C_1 \text{ such that } \phi \text{ is convex, where } \Phi'(t) = \phi(t) \text{ almost everywhere}\} \tag{4}$$

Specifically, Sordo *et al.* (2007) show that

$$I_X(\Phi, z) \leq I_Y(\Phi, z), \text{for all } I(\Phi, z) \in C_2$$

if and only if

$$\int_0^p G_X(t, z) \, dt \leq \int_0^p G_Y(t, z) \, dt, \text{ for all } p \in [0, 1] \text{ and } G_X(1, z) \leq G_Y(1, z). \tag{5}$$

However, even (5) can be a strong requirement for many pair of distributions, which can fail to satisfy it. This justifies the convenience of employing a weaker criterion to compare income distributions in terms of poverty. In this paper, to deal with such situations, a sequence of higher-degree poverty orderings, which generalizes (5), is considered and its normative significance is provided in terms of a family $C_n$ of classes of poverty measures which generalizes (4).

In Section 2, we introduce the family $C_n$ and the $n$degree TIP curve orderings. The main characterization is stated in Section 3. An example is given in Section 4 and Section 5 contains final remarks and conclusions.

## 2. Poverty measures and high-degree poverty orderings

The family $C$ given by functionals of the form (2) contains some important measures. A subclass $S \subset C$ of particular interest emerges from considering the weight function

$$\Phi_n(p) = \left\{ 1 - (1-p)^n \right\}, \, n \geq 1. \tag{6}$$

As noted by Duclos (2000) and Duclos and Grégoire (2002), $I_X(\Phi_n, z)$, $n > 1$, depends upon an ethical parameter $n$, which captures the sensitivity of poverty measurement to "exclusion" or "relative deprivation" aversion: the greater the value of $n$, the more weight is given to the relative deprivation of the poor. They refer to $I_X(\Phi_n, z) = S_X(n, z)$ as the equally distributed equivalent (EDE) poverty gap that is socially equivalent to the actual distribution of poverty gaps and compare its properties with those of additive poverty indices. $S_X(n, z)$ also can be interpreted as the higher poverty gap in a sample of $n$ randomly selected poor individuals. The class $S$ contains some poverty measures that are well known from the literature. It includes the so-called "per-capita income gap" or $FGT(1)$ proposed by Foster *et al.* (1984) and is obtained when $n = 1$ (that is, $\Phi_n(t)$ is the uniform distribution on $(0, 1)$). The Thon (1979), Chakravarty (1983) and Shorrocks (1995) poverty indices are obtained when $n = 2$.

Two more general subclasses of $C$, which turn out to be crucial in the course of this work, are defined below. Let $\Phi^{(i)}$ denote the $i$th derivative of $\Phi$, $i = 1, 2, \ldots$

**Definition 1** $C_n$ *is the class of indices* $I_X(\Phi, z) \in C$ *such that* $\Phi$ *is at least n times differentiable,* $(-1)^i \Phi^{(i+1)} \geq 0$ *for* $i = 0, 1, .., n-1$ *and* $(-1)^{n-1} \Phi^{(n)}$ *is non-increasing[1].* $C_n^*$ *is the class of indices* $I_X(\Phi, z) \in C_n$ *such that* $\Phi^{(i)}(1) = 0$, $i = 1, \ldots, n$.

For $n = 1$ and $n = 2$, $C_n$ reduces to (3) and (4) respectively. Note that $C_{k+1} \subset C_k$ for $k = 1, 2, \ldots$ Also, note that $S_X(k, z) \in C_n^*$ for $k \geq n+1$. On the other hand, that not every measure of interest of $C_n$ belongs to $S$, is shown by the measure proposed by Thon (1983), obtained from (2) by choosing

$$\Phi(t) = \frac{c^2}{4(c-1)} - \frac{1}{c-1} \left( \frac{c}{2} - t \right)^2, \, c > 2.$$

It can be shown, by an argument similar to that used by Duclos (2000), that a social decision-marker who employs $I_X(\Phi, z) \in C_k$, with $k \geq 1$, is more sensitive to transfers occurring within the lower part of the distribution and, as $k$ increases, the weight assigned to the effect of these transfers also increases.

As we follow in the next section, comparisons of income distributions according to the indices of the classes $C_n$ and $C_n^*$, for all integer $n \geq 1$, can be characterized by means

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1.   We also include in $C_n$ indices $I_F(\Phi, z)$ where $\Phi^{(n)}$ exists except possibly at a countable number of points. Thus, $I_F(\Phi_{n,x}, z)$, where $\Phi_{n,x}$ is the $n$iterated integration of a "wedge" function of the form $\Phi_x(t) = (t-x)^+ = \max\{t-x, 0\}$, $x \in (0, 1)$, is included in $C_n$.

of a family of stochastic orderings based on comparing TIP areas and equally distributed equivalent (EDE) poverty gaps. Given a poverty line $z$, denote $G_X^{[1]}(p,z) = G_X(p,z)$, $0 \le p \le 1$, and define

$$G_X^{[n]}(p,z) = \int_0^p G_X^{[n-1]}(t,z)dt, \text{ for } n = 2,3,\dots \text{ and } 0 \le p \le 1. \tag{7}$$

**Definition 2** *Given two income random variables $X$ and $Y$ and a common poverty line $z$, we say that $X$ dominates $Y$ in the nth degree TIP curve ordering (denoted by $X \ge_{TIP(n,z)} Y$) if $S_X(k,z) \ge S_Y(k,z)$ for $k = 1,2,\dots,n$ and $G_X^{[n]}(p,z) \ge G_Y^{[n]}(p,z)$ for all $p \in [0,1]$.*

Before obtaining further results, we need the following easy-to-prove auxiliary lemma.

**Lemma 3** *For any real value $x$, denote $x^+ = \max\{x,0\}$.*
*(i) For a fixed $p \in [0,1]$, the functions $\Psi_{p,1}$, defined by $\Psi_{p,1}(t) = (t-p)^+$ and $\Psi_{p,n}(t)$, defined by*

$$\Psi_{p,n}(t) = \int_0^t \Psi_{p,n-1}(x)dx, \; n = 2,3,\dots, \tag{8}$$

*satisfy $\Psi_{p,n}^{(k)}(t) \ge 0$ for $k = 1,2,\dots,n-1$ and $\Psi_{p,n}^{(n)}(t) \ge 0$ except at $t = p$.*
   *(ii) $I_X(\Phi_{p,n},z)$ belongs to $C_n$, where*

$$\Phi_{p,n}(t) = 1 - \Psi_{p,n}(1-t), \; n = 1,2,\dots \tag{9}$$

We also need the following useful result.

**Lemma 4** *For each $n \ge 2$, we have*

$$G_X^{[n]}(p,z) = \int_0^1 G_X^{[n-k]}(t,z)d\Phi_{1-p,k}(t), \;\; k = 1,2,\dots,n-1. \tag{10}$$

*Proof.* Let $n \ge 2$ fixed. We use induction on $k$ to prove the lemma. For $k = 1$, we have from (9) that

$$\Phi_{1-p,1}(t) = 1 - (p-t)^+ = \begin{cases} 1-p+t & \text{if } t \le p \\ 1 & \text{if } t > p \end{cases}. \tag{11}$$

The right-hand side of (10) equals

$$\int_0^1 G_X^{[n-1]}(t,z)d\Phi_{1-p,1}(t) = \int_0^p G_X^{[n-1]}(t,z)dt,$$

which is $G_X^{[n]}(p,z)$, the left-hand side. For $k = 2$, we have

$$\Phi_{1-p,2}(t) = 1 - \int_0^{1-t}(u-1+p)^+\,du. \tag{12}$$

By using the properties of the Riemann–Stieltjes integral, the right-hand side of (10) equals

$$\int_0^1 G_X^{[n-2]}(t,z)d\Phi_{1-p,2}(t) = \int_0^p (p-t)\,dG_X^{[n-1]}(t,z). \tag{13}$$

Taking account that $G_X^{[n-1]}(0,z) = 0$, integration by parts in (13) yields

$$\int_0^p G_X^{[n-1]}(t,z)\,dt,$$

the left-side hand. Let $k \geq 3$ and assume that the result holds for $k - 1$. It follows from (8) and (9) that

$$\Phi_{1-p,k}(t) = 1 - \int_0^{1-t}\Psi_{1-p,k-1}(u)du$$

and, therefore, the right-hand side of (10) equals

$$\int_0^1 \Psi_{1-p,k-1}(1-t)dG_X^{[n-k+1]}(t,z). \tag{14}$$

Taking into account (9), $\Psi_{1-p,k-1}(0) = 0$ if $k \geq 3$ and $G_X^{[n-k+1]}(0,z) = 0$, integration by parts in (14) yields

$$\int_0^1 G_X^{[n-(k-1)]}(t,z)d\Phi_{1-p,k-1}(t),$$

which is $G_X^{[n]}(p,z)$ by applying the induction hypothesis.                    □

In the following result, we prove that, for each fixed $p \in [0,1]$, $G_X^{[n]}(p,z)$ belongs to $C_n$.

**Theorem 5** *For each fixed $p \in [0,1]$ and $n \geq 1$, $G_F^{[n]}(p,z) \in C_n$.*

Proof. The proof consists in proving that

$$I_X(\Phi_{1-p,n},z) = G_X^{[n]}(p,z) \tag{15}$$

holds for all $n = 1,2,\ldots$ For $n = 1$, using (11), the left-hand side of (15) equals

$$\int_0^1 \left(z - F_z^{-1}(t)\right) d\Phi_{1-p,1}(t) = \int_0^p \left(z - F_z^{-1}(t)\right) dt,$$

which is $G_F^{[1]}(p,z)$, the right-hand side. For $n = 2$, using (12), the left-hand side of (15) equals

$$\int_0^1 \left(z - F_z^{-1}(t)\right) d\Phi_{1-p,2}(t) = \int_0^p (p-t) dG_X^{[1]}(t,z)$$

which is, using integration by parts,

$$\int_0^p G_X^{[1]}(t,z) dt,$$

that is, $G_X^{[2]}(p,z)$, the right-hand side. Let $n \geq 3$. Taking into account (9), the left-hand side of (15) equals

$$\int_0^1 \left(z - F_z^{-1}(t)\right) d\Phi_{1-p,n}(t) = \int_0^1 \Psi_{1-p,n-1}(1-t) dG_X^{[1]}(p,z).$$

Integration by parts and the facts that

$$\Psi_{1-p,n-1}(0) = 0 \text{ for } n \geq 3$$

and

$$G_X^{[1]}(0,z) = 0$$

yield

$$-\int_0^1 G_X^{[1]}(p,z) d\Psi_{1-p,n-1}(1-t) \qquad (16)$$

or, equivalently, using again (9),

$$\int_0^1 G_X^{[1]}(p,z) d\Phi_{1-p,n-1}(t).$$

which is $G_X^{[n]}(p,z)$ by Lemma 4. $\qquad \square$

It is well-known that $S_X(1,z)$ (the per-capita income gap) is $G_X^{[1]}(1,z)$ and $S_X(2,z)$ (the Thon-Chakravarty-Shorrocks indice) is two times the area underneath the curve $G_X^{[1]}(p,z)$. Now, we generalize these results by showing that $S_X(n,z)$ is $n!$ times the area underneath the curve $G_X^{[n-1]}(p,z)$ for all $n \geq 1$.

**Theorem 6** *For all $n \geq 1$, we have $S_X(n,z) = n! G_X^{[n]}(1,z)$.*

*Proof.* From (10) we have

$$G_X^{[n]}(1,z) = \int_0^1 G_X^{[1]}(t,z) d\Phi_{0,n-1}(t). \tag{17}$$

It can be easily shown from (9) that

$$\Phi_{0,n-1}(t) = 1 - \frac{(1-t)^{n-1}}{n-1!}$$

Therefore, integration by parts in (17) yields

$$\frac{1}{n-1!} \int_0^1 (1-t)^{n-1} dG_X^{[1]}(t,z)$$

and this is the same as

$$\frac{1}{n!} \int_0^1 \left(z - F_z^{-1}(t)\right) d\Phi_n(t)$$

where $\Phi_n(t)$ is given by (6), which is $\frac{1}{n!}S_X(n,z)$.                   □

## 3. Characterizations

Now, we characterize the $n$th degree TIP curve dominance in terms of the class $C_n$ defined in Section 2.

**Theorem 7** *Let $X$ and $Y$ be two income random variables. For integers $n \geq 1$, we have*

$$X \geq_{TIP(n,z)} Y \quad \textit{if and only if } I_X(\Phi,z) \geq I_Y(\Phi,z) \textit{ for all } I \in C_n.$$

*Proof.* Necessary condition is immediate since $S_X(k,z) \in C_n$ for $k = 1,2,\ldots,n$ and, from Theorem 5, $G_X^{[n]}(p,z)$ also belongs to $C_n$ for all $p \in [0,1]$. Therefore, by hypothesis, $S_X(k,z) \geq S_Y(k,z)$ for $k = 1,2,\ldots,n$ and $G_X^{[n]}(p,z) \geq G_Y^{[n]}(p,z)$, which means $X \geq_{TIP(n,z)} Y$.

In order to prove the sufficient condition suppose, firstly, that $X \geq_{TIP(1,z)} Y$ and take $I_X(\Phi,z) \in C_1$. Then, $\Phi$ is a concave distribution function on $[0,1]$ and there exists some non-negative, non-increasing and integrable function $\varphi$ such that

$$\Phi(t) = \int_0^t \varphi(x) dx.$$

Therefore,

$$I_X(\Phi, z) = \int_0^1 \left(z - F_z^{-1}(t)\right) d\Phi(t) = \int_0^1 \varphi(t) \, dG_X^{[1]}(t, z).$$

Via integration by parts, we have

$$I_X(\Phi, z) = \varphi(1) S_X(1, z) - \int_0^1 G_X^{[1]}(t, z) d\varphi(t). \tag{18}$$

Since

$$G_X^{[1]}(t, z) \geq G_Y^{[1]}(t, z) \text{ for all } t \in [0, 1]$$

(in particular, $S_X(1, z) \geq S_Y(1, z)$),

$$\varphi(1) \geq 0 \text{ and } d\varphi(t) \leq 0,$$

it follows from (18) that $I_X(\Phi, z) \geq I_Y(\Phi, z)$.

Now, suppose $X \geq_{TIP(n,z)} Y$ and take $I(\Phi, z) \in C_n$, with $n \geq 2$. The first step consists in proving, by induction on $n$, that

$$I_X(\Phi, z) = \sum_{k=1}^{n-2} (-1)^{k+1} \Phi^k(1) S_X(k, z) - \int_0^1 (-1)^{n-1} \Phi^{n-1}(t) \, dG_X^{[n-1]}(t, z). \tag{19}$$

For $n = 2$, (19) is confirmed by using again the properties of the Riemann–Stieltjes integral:

$$I_X(\Phi, z) = \int_0^1 \left(z - F_z^{-1}(t)\right) d\Phi(t) = \int_0^1 \Phi'(t) \, dG_X^{[1]}(t, z),$$

which is the right-hand side of (19). Now suppose inductively that (19) holds for $n$ and show the result holds for $n + 1$. Let $I_X(\Phi, z) \in C_{n+1}$. Note, via integration by parts, that

$$\int_0^1 \Phi^{n-1}(t) \, dG_X^{[n-1]}(t, z) = \Phi^{n-1}(1) S_X(n-1, z) - \int_0^1 G_X^{[n-1]}(t, z) d\Phi^{n-1}(t)$$

which is the same as

$$\Phi^{n-1}(1) S_X(n-1, z) - \int_0^1 \Phi^n(t) \, dG_X^{[n]}(t, z). \tag{20}$$

Since $C_{n+1} \subset C_n$, by the induction hypothesis, $I_F(\Phi, z)$ satisfies (19) and by replacing (20) in (19) we obtain

$$I_X(\Phi, z) = \sum_{k=1}^{n-1} (-1)^{k+1} \Phi^k(1) S_X(k, z) - \int_0^1 (-1)^n \Phi^n(t) \, dG_X^{[n]}(t, z)$$

as required. This proves that (19) holds for all $I_X(\Phi, z) \in C_n$, for all $n \geq 2$. Next, observe that, for $I_X(\Phi, z) \in C_n$, the function

$$\alpha(t) = (-1)^{n-1} \Phi^{n-1}(t) \tag{21}$$

is increasing and concave on $(0, 1)$ and we can write

$$\alpha(t) = \alpha(1) - \int_t^1 \mu(x) dx \tag{22}$$

where $\mu = \alpha'$ (almost everywhere) is non-negative and non-increasing. It is easy to see, by integration by parts, that (22) is the same as writing

$$\alpha(t) = \alpha(1) - \mu(1)(1-t) + \int_0^1 (p-t)^+ d\mu(p). \tag{23}$$

Substitution of (21) into (23) yields

$$(-1)^{n-1} \Phi^{n-1}(t) = (-1)^{n-1} \Phi^{n-1}(1) + (-1)^n \Phi^n(1)(1-t) + \tag{24}$$

$$(-1)^{n+1} \int_0^p (p-t) \, d\Phi^n(p).$$

By substituting (24) into (19) and rearranging terms we have that

$$I_X(\Phi, z) = \sum_{k=1}^{n-1} (-1)^{k+1} \Phi^k(1) S_X(k, z) + \tag{25}$$

$$(-1)^{n+1} \Phi^n(1) \int_0^1 (1-t) \, dG_X^{[n-1]}(t, z) +$$

$$(-1)^n \int_0^1 \int_0^p (p-t) \, dG_X^{[n-1]}(t, z) d\Phi^n(p).$$

(Fubini's Theorem has been applied in the last term). Since

$$\int_0^p (p-t) \, dG_X^{[n-1]}(t, z) = G_X^{[n]}(p, z)$$

and, consequently,

$$\int_0^1 (1-t) \, dG_X^{[n-1]}(t, z) = S_X(n, z),$$

(25) can be rewritten as follows:

$$I_X(\Phi,z) = \sum_{k=1}^{n}(-1)^{k+1}\Phi^k(1)S_X(k,z) + (-1)^n\int_0^1 G_X^{[n]}(p,z)d\Phi^n(p). \qquad (26)$$

We complete the proof by noting that

$$(-1)^{k+1}\Phi^k(1) \geq 0 \text{ for } k=1,\ldots,n,$$

$$S_X(k,z) \geq S_Y(k,z), \text{ for } k=1,\ldots,n,$$

$$G_X^{[n]}(p,z) \geq G_Y^{[n]}(p,z) \text{ for all } p \in [0,1]$$

and

$$(-1)^n d\Phi^n(p) \geq 0. \qquad \square$$

If we restrict attention to the class $C_n^*$, then a comparison of $n$TIP curves is enough to obtain a characterization. The proof of the next result follows easily from (26).

**Corollary 8** *Let X and Y be two income random variables. For integers $n \geq 1$, we have*

$$G_X^{[n]}(p,z) \geq G_Y^{[n]}(p,z) \text{ for all } p \in [0,1]$$

*if and only if*

$$I_X(\Phi,z) \geq I_Y(\Phi,z) \text{ for all } I \in C_n^*.$$

## 4. An example

It is well-known that empirical income distribution data fit well to lognormal form (see, for example, Harrison (1981) and Cowell (1999)). Moreover, the use of the lognormal model is "probably the most standard approximation of empirical data distributions in the applied literature" (Bourguignon, 2003, page 11). See Lambert (2009) and references therein for applications of this model in poverty analysis. Recall that a lognormal random variable $X$ has a density function of the form

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}, \quad x > 0, \quad \sigma > 0, \ \mu \in \mathbb{R}$$

and the mean and the standard deviation are given, respectively, by $E[X] = \exp\left\{\mu + \frac{\sigma^2}{2}\right\}$ and $SD(X) = \sqrt{\left(e^{\sigma^2} - 1\right) e^{2\mu + \sigma^2}}$.

In order to illustrate the applicability of the comparison method proposed in this paper, we have simulated two samples with sizes $n = m = 100$ from two underlying lognormal distributions $X$ and $Y$, with respective means $E[X] = 9030$ and $E[Y] = 9010$ and standard deviations $SD(X) = SD(Y) = 3100$. The reference poverty line is set at $z = 6500$ (since $F_X(z) = 0.21$ and $F_Y(z) = 0.19$, this choice appears to be a reasonable poverty line for poverty comparisons between these models[2]).

In order to compare the poverty associated to these income distributions, we start by comparing the corresponding "per-capita income gaps" $S_X(1, z)$ and $S_Y(1, z)$, which represent the sum of the poverty gaps of the poor. The evaluation of these indices with DAD 4.5 (a programme freely distributed by Duclos *et al.* (2006), designed to facilitate the analysis of social welfare, inequality and poverty), gives $S_X(1, z) = 236.67$ and $S_Y(1, z) = 234.57$ and we can say that poverty, as measured by this index, is greater in $X$ than in $Y$. However, "any choice of a single measure is apt to be arbitrary" (Foster, 1984, page 242), and different choices may produce different conclusions. We reduce this arbitrariness by considering a broader class of poverty measures than $S(1, z)$, given by

$$C_1 = \{I(\Phi, z) \text{ of the form (2) such that } \Phi \text{ is concave}\}.$$

Each member of $C_1$ is interpreted as a weighted sum of the poverty gaps of the poor. Obviously, it is impossible to check poverty orderings for all measures in $C_1$ and we prefer to plot the corresponding TIP curves $G_X^{[1]}(p, z)$ and $G_X^{[1]}(p, z)$. Following Duclos and Araar (2006, Section 10.1), non-intersection of these curves is equivalent to the unanimous ordering generated by the class $C_1$. Unfortunately, Figure 1 shows that the TIP curves cross twice (the first is at around $p = 0,11$ and the second is at around $p = 0,19$), therefore the inequality $I_X(\Phi, z) \leq I_Y(\Phi, z)$ fails to be satisfied for some member of $C_1$. In other words, the comparison between $X$ and $Y$ in terms of poverty measures in $C_1$ is ambiguous.

Fortunately, as we have shown in Section 3, an unambiguous ordering between $X$ and $Y$ is still possible by focussing on a subclass of $C_1$ and moving from the first degree TIP ordering to the second degree TIP ordering (and, more generally, to the $n$-degree TIP ordering, $n \geq 2$). The second degree TIP ordering requires the evaluation of $S_X(k, z)$ and $S_Y(k, z)$ for $k = 1, 2$, and the comparisons of the curves $G_X^{[2]}(p, z)$ and $G_Y^{[2]}(p, z)$. The evaluation of $S_X(2, z)$ and $S_Y(2, z)$ with DAD 4.5 gives $S_X(2, z) = 444.75$ and $S_Y(2, z) = 439.71$. Therefore, we have

$$S_X(1, z) > S_Y(1, z) \text{ and } S_X(2, z) > S_Y(2, z).$$

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2. In Spain, for example, the percentage of persons below the poverty line is 19.6% (Quality of Life Survey, 2008, I.N.E.)
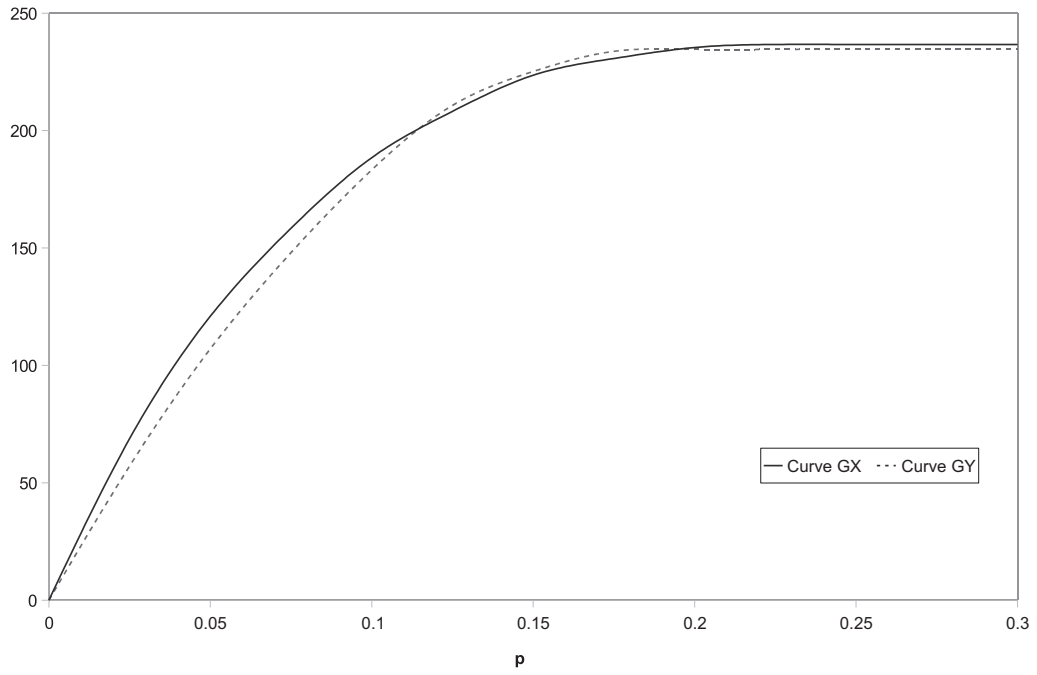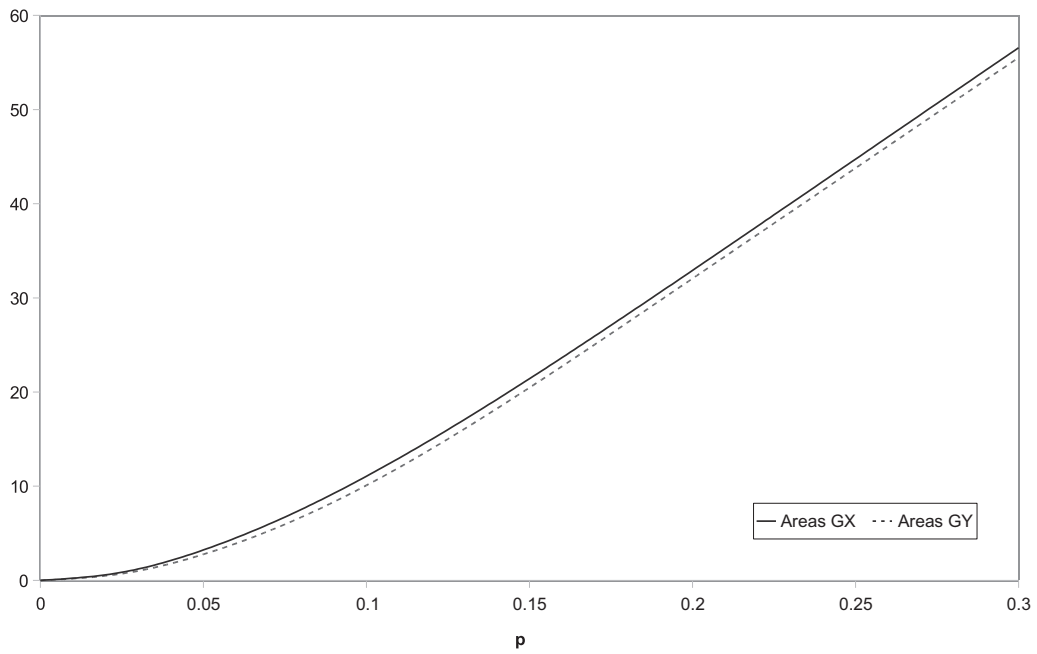
**Figure 1:** $(p, G^{[1]}(p, z))$



**Figure 2:** $(p, G^{[2]}(p, z))$

Moreover, Figure 2 shows that $G_X^{[2]}(p,z)$ is above $G_Y^{[2]}(p,z)$ for all $p$ in $(0,1)$ (the curves are plotted up to $p = 0.3$; this is due to the fact that the second cross between the TIP curves is at around $p = 0.19$; therefore, from this $p$ on, $G_X^{[2]}(p,z)$ is everywhere above $G_Y^{[2]}(p,z)$). Thus, $X \geq_{TIP(2,z)} Y$ holds and from Theorem 7 it follows that $I_X(\Phi, z) \geq I_Y(\Phi, z)$ for all measures in

$$C_2 = \left\{ I(\Phi, z) \in C_1 \text{ such that } \Phi' \text{ is convex} \right\}.$$

We conclude this illustration by noting that increasing the degree of dominance (moving from the first degree TIP ordering to the second degree TIP ordering) makes poverty in $X$ unambiguously larger than in $Y$. Since any index of $C_2$ not belonging to $C_1$ is more sensitive to the distribution of income among the poorest, this is equivalent to saying that poverty in $X$ is unambiguously larger than in $Y$ when sufficient weight is given to the effect of income changes among the bottom of the distribution.

## 5. Final remarks

In this paper, we have tried to advance in obtaining comparable poverty results when TIP curves intersect, by considering a sequence of dominance criteria (the $n$degree TIP curve dominance) based on TIP areas and $S$-indices. The normative meaning of these criteria has been provided in terms of a class $C_n$ of linear rank-based poverty measures with the property that, the larger the value of $n$, the greater the weight assigned to the effect of income changes among the bottom of the distribution.

Duclos and Grégoire (2002) have shown that the properties of the $S$-indices compare rather well with those of the $FGT$ (Foster *et al.*, 1984) additive indices. Some results in this work confirm this conclusion. Given an income distribution $F$, a poverty line $z$ and a non-negative integer $\alpha$, the $FGT(\alpha)$ index is defined by

$$FGT_\alpha(F,z) = \int_0^1 (z-x)^\alpha \, dF(x).$$

Foster and Shorrocks (1998a) note that

$$FGT_\alpha(F,z) = \alpha! F_{\alpha+1}(z) \text{ for all } z, \tag{27}$$

where $F_1(x) = F(x)$, $F_k(x) = \int_0^x F_{k-1}(t) \, dt$, $k = 1, 2, ..$ and provide the following link between the poverty order induced by $FGT(\alpha)$ for all $z \in (0, \infty)$ and the $(\alpha+1)$th degree stochastic dominance:

$$FGT_\alpha(F,z) \geq FGT_\alpha(H,z) \; \forall \, z \in (0, \infty) \Longleftrightarrow F_{\alpha+1}(z) \geq H_{\alpha+1}(z) \, \forall \, z \in (0, \infty)$$

The relation

$$S_X(n,z) = n! G_X^{[n]}(1,z) \tag{28}$$

stated in Theorem 6 is somewhat similar to (27) and suggests that the role played by $S_X(n,z)$ in the dual approach (in the sense of Duclos and Araar, 2006) is as important as the role of $FGT$ indices in the primal one. The characterization

$$S_X(n,z) \geq S_Y(n,z) \forall z \in (0,\infty) \iff G_X^{[n]}(1,z) \geq G_Y^{[n]}(1,z) \forall z \in (0,\infty)$$

(which follows from Theorem 6) shows that $X$ has unambiguously more poverty that $Y$ with respect to the poverty measure $S(n,z)$ for all $z \in (0,\infty)$ if, and only if, the area underneath the curve $G_X^{[n-1]}(p,z)$ is bigger than the area underneath $G_Y^{[n-1]}(p,z)$ for all $z \in (0,1)$. (28) also reveals an important interrelationship among poverty-line orderings by different members of $S$. Since

$$G_X^{[n]}(1,z) \geq G_Y^{[n]}(1,z) \implies G_X^{[k]}(1,z) \geq G_Y^{[k]}(1,z) \text{ for } k \geq n$$

it follows from (28) that

$$S_X(n,z) \geq S_Y(n,z) \forall z \implies S_X(k,z) \geq S_Y(k,z) \forall z, \forall k \geq n.$$

## Aknowledgement

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