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Statistical modelling and forecasting of outstanding liabilities in non-life insurance

María Dolores Martínez-Miranda[1,2], Jens Perch Nielsen[2]
and Mario V. Wüthrich[3]

Abstract

Non-life insurance companies need to build reserves to meet their claims liability cash flows. They often work with aggregated data. Recently it has been suggested that better statistical properties can be obtained when more aggregated data are available for statistical analysis than just the classical aggregated payments. When also the aggregated number of claims is available one can define a full statistical model of the nature of the number of claims, their delay until payment and the nature of these payments. In this paper we provide a new development in this direction by entering yet another set of aggregated data, namely the number of payments and when they occurred. A new element of our statistical analysis is that we are able to incorporate inflationary trends of payments in a direct and explicit way. Our new method is illustrated on a real life data set.

MSC: 62P05 Applications to actuarial sciences and financial mathematics, 91B30 Risk theory, insurance.

Keywords: Outstanding loss liabilities, claims settlement process, claims reserving, chain ladder method, individual claims data, prediction uncertainty, bootstrap, early warning systems.

1. Introduction

Non-life insurance companies need to forecast future payments arising from claims where the companies already received the insurance premium. The discounted aggregate of these future payments is called the reserve (outstanding liabilities) and is one of the most important components in the accounts of a non-life company. The reserve is most often set by actuaries and the reserving problem is omnipresent in the literature of actuarial science. However, the history of the reserving problem is not a mathematical

[1] University of Granada, Department of Statistics and O.R., Spain.
[2] Cass Business School, City University, London, UK.
[3] ETH Zurich, RiskLab, Department of Mathematics, Zurich, Switzerland.
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statistical history even though it clearly is a mathematical statistical forecasting problem. The history is a practical one, where actuaries have had to develop methodologies to set reserves at a time when mathematical statistics was not well developed. The most popular reserving method used by almost all insurance companies is called the chain ladder method by actuaries. Most practical actuaries would talk about chain ladder as a method rather than as a mathematical statistical model even though the actuarial literature has shown a close connection between the chain ladder method and the multiplicative Poisson model. It was only just recently that this multiplicative Poisson model was identified as belonging to the class of exponential families implying well defined solutions to the maximum likelihood estimators and it was also only recently that the explicit expressions of the entering parameters were derived, see Kuang, Nielsen and Nielsen (2009). While practical actuaries work with chain ladder forecasts identical to the forecasts provided by a multiplicative Poisson model, they do not work with the distributional properties of the multiplicative Poisson model. Other distributional properties are preferred, often based on *ad hoc* bootstrap type of procedures. In this paper we build on theory recently derived in three interconnected papers. The main underlying idea of these three papers is that more data (aggregated reported number of claims) should be added to classical actuarial data to allow for a better and more precise formulation of the underlying mathematical statistical model driving the claims development process defining the reserve. The first of these papers (Verrall, Nielsen and Jessen, 2010) defines the simplest possible version of such a model, the second (Martínez-Miranda, Nielsen, Nielsen and Verrall, 2011) develops a bootstrap methodology to assess the distribution of such a model, but the most important of these three papers is perhaps the third one (Martínez-Miranda, Nielsen and Verrall, 2012). This paper shows that a slight modification of Verrall *et al.* (2010)'s model, with one particular moment type estimation method, provides us with a well-defined mathematical statistical model exactly replicating the reserving estimates one would obtain using the classical chain ladder method. This model has trustable distributional properties that can be used in practice by actuaries. In this paper we take the ideas of the above three papers one step further and add yet another piece of data (aggregated number of payments) to our data set and we show that important insights of the claim development process result when incorporating this extra piece of information in our mathematical statistical model. We follow in this paper Martínez-Miranda *et al.* (2012) and work with moment type of estimators. Our hope is that this paper provides information to the mathematical statistician wishing to use their excellent tools on this important real life problem and can perhaps be helpful in bringing mathematical statisticians into this important field. The notation and vocabulary of this paper are deliberately closely related to classical actuarial terminology while describing a well-defined mathematical statistical model. This is a deliberate attempt to bridge the gap between classical actuarial terminology, often obscure to mathematical statisticians, and standard mathematical statistical model formulations that might seem unrelated to classical reserving for many actuaries.

The general post credit crunch atmosphere in the financial sector emphasizes a better understanding of outstanding loss liabilities of non-life insurance companies, with reserving models as one of the essential technical building blocks. However, the insurance industry is also gaining new territory in new markets, where better early warning reserving systems are required than that provided by the old chain ladder methodology. In this paper we introduce a new reserving methodology with an automatic early warning system to detect important irregularities in the claims development process. Our methodology requires more detailed data than classical reserving methods. The point of view taken is that the aggregated payments do not provide us with sufficient mathematical statistical information, we argue that also the number of payments and the number of reported claims are needed. This enables us to embed a variety of new claims inflation type of information in our overall model. We consider severity inflation, underwriting year inflation and claims delay inflation and show how to incorporate those in the reserving process. The calendar inflation is not treated in detail in this paper, but it could have been extracted up front using the Kuang, Nielsen and Nielsen (2011) methodology of calendar inflation (see also Kuang, Nielsen and Nielsen 2008a,b) .

In the next section we define the model on the micro-level. The basis of our model is the compound Poisson processes studied in Norberg (1993, 1999) and Jessen, Mikosch and Samorodnitsky (2011). We show how we need to structure these compound Poisson processes on the micro-level so that we obtain a chain ladder claims reserving method on the aggregate level. Such connection is proved from first moments calculations which are provided in Appendix A. In Section 3 we provide estimates of the parameters of the model. From the estimated model, point forecasts for the reserve are given in Section 4. Using bootstrap methods we provide in Section 6 (together with Appendix B) an approximation of the full predictive distribution of the outstanding loss liabilities. The methods proposed in this paper are illustrated using a dataset from the insurance industry, given in Appendix C. The focus of this application is to provide an estimate of the claims reserves and to detect irregularities in the data.

2. Model setup

2.1. Data and micro-level structure

In classical reserving methods the data upon which projections of future claims are usually represented by so called run-off triangles. This format tabulates the claim data (payments, numbers of reported or paid claims, etc.) according to the period in which the claim arose (called underwriting or accident period) and the period in which the payment (or other action) was made. The difference between the payment period and the accident period is referred to as the development period. The data are usually aggregated in years or quarters of years, but other time periods can also be used depending on the business line. Hereafter we write years as the periods considered in the aggregation. We denote accident years by $i = 1, \ldots, m$, and development years by $j = 0, \ldots, m-1$, where $m \in \mathbb{N}$ denotes the last observed accident year. Then the available data lie in the triangle $\mathcal{I}_m = \{(i,j);\ i = 1, \ldots, m; j = 0, \ldots, m-1; i+j \leq m\}$. In Appendix C we show an example of this type of data.

The methods proposed in this paper consider such run-off triangles as input data. In fact we will need more triangles to provide a more precise formulation of the mathematical statistical micromodel that underlies the claims development process defining the reserve. It is a parametric model that is deliberately formulated in such a way that the entering parameters are identifiable and estimable from three aggregated data sets: number of reported claims, number of payments and aggregated paid amounts. These stochastic variables are described in the following:

- Number of reported claims of accident year i with a reporting delay of j years, denoted by $N_{i,j}$.
- Number of payments. Each of these $N_{i,j}$ reported claims generates a claims payment cash flow. We denote by $R_{i,j,l}$ the number of payments generated by these $N_{i,j}$ reported claims that have a payment delay of $l \geq 0$ years. That is, $R_{i,j,l}$ is the number of payments in accounting year $i+j+l$ for claims that have occurred in accounting year i and were reported in accounting year $i+j$.
- Individual claims payments. Each of these $R_{i,j,l}$ claims payments has size $Y^{(k)}_{i,j,l}$, for $k = 1, \ldots, R_{i,j,l}$.

Often, claims payment data is not available on the micro-level structure described by $\{N_{i,j};\ (i,j) \in \mathcal{I}_m\} \cup \{R_{i,j,l}, Y^{(k)}_{i,j,l};\ (i,j) \in \mathcal{I}_m,\ i+j+l \leq m,\ k \geq 1\}$. Therefore, we define the following aggregate claims payment information. The total number of payments in accounting year $i+j$ from claims with accident year i is given by

$$R_{i,j} = \sum_{l=0}^{j} R_{i,j-l,l}. \tag{1}$$

These $R_{i,j}$ are the number of payments in accounting year $i+j$ generated by all claims with accident year i which where reported prior to (and including) accounting year $i+j$, i.e. these are payments from the $N_{i,j-l}$ reported claims, with $l = 0, \ldots, j$. The payments (total quantity paid) in accounting year $i+j$ from claims with accident year i are then given by

$$X_{i,j} = \sum_{l=0}^{j} \sum_{k=1}^{R_{i,j-l,l}} Y^{(k)}_{i,j-l,l}. \tag{2}$$

From these definitions we assume that the available information at time m consists of the following three σ-fields (upper claims development triangles):

$$\mathcal{N}_m = \sigma\left\{N_{i,j};\ (i,j) \in \mathcal{I}_m\right\},$$
$$\mathcal{R}_m = \sigma\left\{R_{i,j};\ (i,j) \in \mathcal{I}_m\right\},$$
$$\mathcal{X}_m = \sigma\left\{X_{i,j};\ (i,j) \in \mathcal{I}_m\right\},$$

and the aim is to predict the total payments in the future:

$$\mathcal{X}_m^c = \left\{X_{i,j};\ (i,j) \in \mathcal{J}_m\right\},$$

where $\mathcal{J}_m = \{(i,j); i = 2,\dots,m,\ j = 0,\dots,m-1,\ i+j > m\}$ is the lower (inexperienced) triangle.

Classical reserving methods as the chain ladder method provide predictions for $\mathcal{X}_m^c$. However, a better description of the reserving problem would be provided if we are able to separate these future payments in the lower triangle into payments for claims that have been already reported (prior to and including accounting year m) and claims that will be reported after accounting year m. The first class of claims is contained in the number of reported claims $\mathcal{N}_m$, and constitutes what is called the reported but not settled (RBNS) claims reserves. The latter class contains the so-called incurred but not reported (IBNR) claims and constitutes the IBNR claims reserves. Such a distinction is often important, for example, in the calculation of unallocated loss adjustment expenses (ULAE), see Wüthrich, Bülmann and Furrer (2010, Section 5.6). If we apply the classical chain ladder method then we predict $\mathcal{X}_m^c$ based solely on the information $\mathcal{X}_m$, thus, we predict the outstanding loss liabilities on a rather aggregate level, which does not allow a distinction between RBNS and IBNR claims reserves.

2.2. Model assumptions

With the above definitions we assume the following hypotheses about the micro-level structure.

(A1) All random variables in different accident years $i \in \{1,\dots,m\}$ are independent.

(A2) The numbers of reported claims $N_{i,0},\dots,N_{i,m-1}$ are independent and Poisson distributed with cross-classified means $\mathbb{E}[N_{i,j}] = \vartheta_i\beta_j$, for given parameters $\vartheta_i > 0$, $\beta_j > 0$ with normalization $\vartheta_1 = 1$.

(A3) The claims payments

$$X_{i,j,l} = \sum_{k=1}^{R_{i,j,l}} Y_{i,j,l}^{(k)}$$

are, conditionally given $N_{i,0},\dots,N_{i,m-1}$, independent (in $l \geq 0$) and compound Poisson distributed with

- $R_{i,j,l}|_{\{N_{i,0},\ldots,N_{i,m-1}\}} \sim \text{Poi}(N_{i,j}\pi_l)$ with given parameter $\pi_l > 0$;
- $Y_{i,j,l}^{(k)}|_{\{N_{i,0},\ldots,N_{i,m-1}\}} \stackrel{(d)}{=} Y_{i,j,l}^{(k)}$ are i.i.d. for $k \geq 1$ with the first two moments given by

$$\mathbb{E}\left[Y_{i,j,l}^{(1)}\right] = \nu_i\, \mu_{j,l} \qquad \text{and} \qquad \mathbb{E}\left[\left(Y_{i,j,l}^{(1)}\right)^2\right] = \nu_i^2\, s_{j,l}^2,$$

for parameters $\nu_i, \mu_{j,l}, s_{j,l} \in \mathbb{R}_+$ with normalization $\nu_1 = 1$.

One crucial point in assumption (A3) is that the claim size (or severity) distribution of $Y_{i,j,l}^{(k)}$ can be split into an accident year dependent part ν_i which models claims inflation in the accident year direction, and a development year dependent part $\mu_{j,l}$ which takes care of reporting delay $j \geq 0$ and payment delay $l \geq 0$. Note that assumption (A3) implies that the payments $Y_{i,j,l}^{(k)}$ are independent from the number of reported claims $N_{i,j}$ as well as from the number of payments $R_{i,j,l}$ (conditional compound Poisson model assumption).

The choices $\vartheta_1 = \nu_1 = 1$ will make the parameters identifiable in the estimation procedure. One can also use other normalizations, such as e.g. $\sum_j \beta_j = 1$ (normalized claims reporting pattern). However, our choice is rather simple to implement and other normalizations are obtained by rescaling.

3. Parameter estimation

The estimation of the model parameters, $\{\vartheta_i, \beta_j, \pi_l, \nu_i, \mu_{j,l}; i = 1, \ldots, m, j, l = 0, \ldots, m-1\}$, can be solved just using the simple chain ladder method on the three input triangles. The only requirement is to demonstrate that the random variables $N_{i,j}$, $R_{i,j}$ and $X_{i,j}$ all have the same cross-classified mean structure, which is the chain ladder mean structure. As was discussed in Martínez-Miranda *et al.* (2012) this can be done from model specifications about just the first moment of the underlying stochastic components. Further purposes about deriving the distribution of the future payments requires conditions on higher order moments and also a more detailed specification including distributional assumptions (see Martínez-Miranda *et al.* 2012 for further explanation). Under the distributional model proposed here, we suggest in Section 6 an estimator for the second moment parameters $s_{j,l}$ $(j, l = 0, \ldots, m-1)$ to derive then the predictive distribution.

Therefore we next provide estimates of the parameters based in the first moment of the random variables, $N_{i,j}$, $R_{i,j}$ and $X_{i,j}$. We have deferred such calculations to Appendix A in order to facilitate the reading of the paper. Specifically in Propositions 2 and 3 we have obtained that the first moments of the three sets of random variables $N_{i,j}$, $R_{i,j}$ and $X_{i,j}$ all have the same cross-classified mean structure. Also we have established connections among the parameters in the model through the following equations:

$$\alpha_i = \vartheta_i \ \nu_i, \tag{3}$$

$$\lambda_j = \sum_{l=0}^{j} \beta_{j-l} \ \pi_l, \tag{4}$$

$$\gamma_j = \sum_{l=0}^{j} \beta_{j-l} \ \pi_l \ \mu_{j-l,l}, \tag{5}$$

From these initial steps our aim is to estimate the corresponding parameters based on the information in $\mathcal{N}_m$, $\mathcal{R}_m$ and $\mathcal{X}_m$, and by applying the simple chain ladder method to each triangle. As an example, we demonstrate the estimation for the observed number of reported claims $\mathcal{N}_m$ and the parameters ϑ_i and β_j. The remaining parameters are estimated in the same way, but based on $\mathcal{R}_m$ and $\mathcal{X}_m$, respectively. In a distribution-free approach we rely on moment estimators. If we aggregate rows and columns, respectively, over the set of information $\mathcal{I}_m$ we obtain the first moment equalities

$$\sum_{k=0}^{m-i} \mathbb{E}[N_{i,k}] \ = \ \vartheta_i \sum_{k=0}^{m-i} \beta_k \qquad \text{for } i = 1, \ldots, m, \tag{6}$$

$$\sum_{k=1}^{m-j} \mathbb{E}[N_{k,j}] \ = \ \beta_j \sum_{k=1}^{m-j} \vartheta_k \qquad \text{for } j = 0, \ldots, m-1. \tag{7}$$

Unbiased estimators for the right-hand side of these equalities are obtained by replacing the moments $\mathbb{E}[N_{i,j}]$, $(i,j) \in \mathcal{I}_m$, by their observations $N_{i,j} \in \mathcal{N}_m$. Then the resulting system of linear equations is solved for ϑ_i and β_j, which provides the corresponding estimators for these parameters. This is in the spirit of the "total marginals" method of Bailey (1963) and Jung (1968). Kremer (1985) and Mack (1991) have shown that in the case of triangular data $\mathcal{N}_m$ this leads to the chain ladder estimators that can be calculated in closed form. Thus,

$\mathcal{N}_m$ provides the chain ladder estimators $\widehat{\vartheta}_i^{(1)}$ and $\widehat{\beta}_j$ for ϑ_i and β_j,
$\mathcal{R}_m$ provides the chain ladder estimators $\widehat{\vartheta}_i^{(2)}$ and $\widehat{\lambda}_j$ for ϑ_i and λ_j,
$\mathcal{X}_m$ provides the chain ladder estimators $\widehat{\alpha}_i$ and $\widehat{\gamma}_j$ for α_i and γ_j,

with $\widehat{\vartheta}_1^{(1)} = \widehat{\vartheta}_1^{(2)} = \widehat{\alpha}_1 = 1$ (initialization in cross-classified means). Note that we obtain two different estimators $\widehat{\vartheta}_i^{(1)}$ and $\widehat{\vartheta}_i^{(2)}$ for the same parameter ϑ_i. However, their values should not be too different, otherwise this indicates that the model may not fit to the claims reserving problem. In order to estimate ϑ_i we could now take a credibility weighted average between $\widehat{\vartheta}_i^{(1)}$ and $\widehat{\vartheta}_i^{(2)}$. For simplicity we set $\widehat{\vartheta}_i$ as the arithmetic mean between $\widehat{\vartheta}_i^{(1)}$ and $\widehat{\vartheta}_i^{(2)}$. Anyway, the appropriateness of this choice should always be checked on the data. Using equality (3) we can estimate the accident year inflation parameter ν_i by

$$\widehat{\nu}_i = \widehat{\alpha}_i / \widehat{\vartheta}_i \qquad \text{for } i = 1, \dots, m. \tag{8}$$

Thus, it remains to estimate the parameters π_l and $\mu_{j,l}$ $(j, l = 0, \dots, m-1)$. There are different ways to estimate these parameters. We start with π_l using the equality (4). If we rewrite this equation in vector notation we have

$$(\lambda_0, \dots, \lambda_{m-1})^\mathsf{T} \;=\; B_\beta \, (\pi_0, \dots, \pi_{m-1})^\mathsf{T},$$

for an appropriate matrix $\mathbf{B}_\beta = \mathbf{B}_{\beta_0,\dots,\beta_{m-1}} \in \mathbb{R}^{m\times m}$. This matrix is estimated by $\widehat{\mathbf{B}}_\beta = \mathbf{B}_{\widehat{\beta}_0,\dots,\widehat{\beta}_{m-1}} \in \mathbb{R}^{m\times m}$ and then we can provide estimates, $\widehat{\pi}_0, \dots, \widehat{\pi}_{m-1}$, by solving the following system:

$$(\widehat{\pi}_0, \dots, \widehat{\pi}_{m-1})^\mathsf{T} \;=\; \widehat{\mathbf{B}}_\beta^{-1} \left(\widehat{\lambda}_0, \dots, \widehat{\lambda}_{m-1}\right)^\mathsf{T}. \tag{9}$$

The estimation of $\mu_{j,l}$ needs more care because the model is over-parametrized. In order to reduce the number of parameters we make one of the following two assumptions

$$\mu_{j,l} \equiv \mu_l \tag{10}$$

or

$$\mu_{j,l} \equiv \mu_j. \tag{11}$$

Using the condition (10) and the equality (5) we have that

$$(\gamma_0, \dots, \gamma_{m-1})^\mathsf{T} \;=\; \mathbf{B}_\beta \, (\pi_0\mu_0, \dots, \pi_{m-1}\mu_{m-1})^\mathsf{T},$$

for matrix $\mathbf{B}_\beta = \mathbf{B}_{\beta_0,\dots,\beta_{m-1}} \in \mathbb{R}^{m\times m}$. If this matrix is again estimated by $\widehat{\mathbf{B}}_\beta = \mathbf{B}_{\widehat{\beta}_0,\dots,\widehat{\beta}_{m-1}}$ we obtain estimates $\widehat{\pi\mu}_0, \dots, \widehat{\pi\mu}_{m-1}$ as the solution of the following system:

$$\left(\widehat{\pi\mu}_0, \dots, \widehat{\pi\mu}_{m-1}\right)^\mathsf{T} \;=\; \widehat{\mathbf{B}}_\beta^{-1} (\widehat{\gamma}_0, \dots, \widehat{\gamma}_{m-1})^\mathsf{T}, \tag{12}$$

and, finally, the estimator for $\mu_{j,l}$ assumption (10) is given by $\widehat{\mu}_{j,l} = \widehat{\mu}_l = \widehat{\pi\mu}_l / \widehat{\pi}_l$.

On the other hand, using assumption (11) and rewriting (5) we have the following system

$$(\gamma_0, \dots, \gamma_{m-1})^\mathsf{T} \;=\; \mathbf{B}_\pi \, (\beta_0\mu_0, \dots, \beta_{m-1}\mu_{m-1})^\mathsf{T},$$

for matrix $\mathbf{B}_\pi = \mathbf{B}_{\pi_0,\dots,\pi_{m-1}} \in \mathbb{R}^{m\times m}$. And again plugging in the estimated matrix $\widehat{\mathbf{B}}_\pi = \mathbf{B}_{\widehat{\pi}_0,\dots,\widehat{\pi}_{m-1}} \in \mathbb{R}^{m\times m}$, we obtain the estimates, $\widehat{\beta\mu}_0, \dots, \widehat{\beta\mu}_{m-1}$, by solving the system

$$\left(\widehat{\beta\mu}_0, \dots, \widehat{\beta\mu}_{m-1}\right)^\mathsf{T} \;=\; \widehat{\mathbf{B}}_\pi^{-1} (\widehat{\gamma}_0, \dots, \widehat{\gamma}_{m-1})^\mathsf{T}. \tag{13}$$

This yields the estimator $\widehat{\mu}_{j,l} = \widehat{\mu}_j = \widehat{\beta\mu}_j / \widehat{\beta}_j$.

The above procedure provides estimates for all the parameters required for point prediction purposes, under the additional assumption (10) or (11). In the next section we are going to describe how they are used to predict the outstanding loss liabilities $\mathcal{X}_m^c$ at time m. Moreover, we will also discuss further adjustments to these estimators in practise.

4. Point forecasts

Point predictions for the outstanding loss liabilities can be derived as estimated unconditional (or conditional) means of the aggregated payments, $X_{i,j}$, in the lower triangle, $\mathcal{J}_m$. In the previous section we have estimated all the parameters in the model from the observations $\mathcal{N}_m$, $\mathcal{R}_m$ and $\mathcal{X}_m$. It only remains to estimate the second moment parameters $s_{j,l}$ $(j,l = 0,\dots,m-1)$ of the size of the individual payments. But, as we pointed in the previous section, such higher order moments are not involved in the point forecasts. Therefore, we have all that is necessary to predict the outstanding liabilities, $\mathcal{X}_m^c$. At time m the conditionally expected outstanding loss liability cash flows in $\mathcal{X}_m^c$ are given by

$$Z_m = \sum_{i=2}^{m} \sum_{j=m-i+1}^{m-1} \mathbb{E}\left[X_{i,j} | \mathcal{N}_m, \mathcal{R}_m, \mathcal{X}_m\right].$$

If we only rely on the observations $\mathcal{X}_m$, then we can only estimate the parameters α_i and γ_j. Thus, in this case we set

$$\widehat{Z}_m^{CL} = \sum_{i=2}^{m} \sum_{j=m-i+1}^{m-1} \widehat{\alpha}_i \, \widehat{\gamma}_j,$$

which provides an estimator for Z_m. The crucial property of this estimator $\widehat{Z}_m^{CL}$ is that it provides the chain ladder reserves exactly (see Kremer 1985, Mack 1991 and Section 2.4 in Wüthrich and Merz 2008). Having additional information $\mathcal{N}_m$ and $\mathcal{R}_m$ we can refine this estimate. We have

$$\begin{aligned} Z_m &= \sum_{i=2}^{m} \sum_{j=m-i+1}^{m-1} \sum_{l=0}^{j} \mathbb{E}\left[\sum_{k=1}^{R_{i,j-l,l}} Y_{i,j-l,l}^{(k)} \middle| \mathcal{N}_m, \mathcal{R}_m, \mathcal{X}_m \right] \\ &= \sum_{i=2}^{m} \sum_{j=m-i+1}^{m-1} \sum_{l=i+j-m}^{j} \mathbb{E}\left[\sum_{k=1}^{R_{i,j-l,l}} Y_{i,j-l,l}^{(k)} \middle| \mathcal{N}_m, \mathcal{R}_m, \mathcal{X}_m \right] \\ &\quad + \sum_{i=2}^{m} \sum_{j=m-i+1}^{m-1} \sum_{l=0}^{i+j-m-1} \mathbb{E}\left[\sum_{k=1}^{R_{i,j-l,l}} Y_{i,j-l,l}^{(k)} \middle| \mathcal{N}_m, \mathcal{R}_m, \mathcal{X}_m \right]. \end{aligned}$$

Note that the decoupling separates RBNS and IBNR claims: if $i+j-l \leq m$ then the payment $Y_{i,j-l,l}^{(k)}$ belongs to a claim that has been reported prior to (and including) accounting year m, and henceforth is an RBNS claim at time m. Therefore, we define

$$Z_m^{\text{RBNS}} = \sum_{i=2}^{m} \sum_{j=m-i+1}^{m-1} \sum_{l=i+j-m}^{j} \mathbb{E}\left[\sum_{k=1}^{R_{i,j-l,l}} Y_{i,j-l,l}^{(k)} \middle| \mathcal{N}_m, \mathcal{R}_m, \mathcal{X}_m \right],$$

$$Z_m^{\text{IBNR}} = \sum_{i=2}^{m} \sum_{j=m-i+1}^{m-1} \sum_{l=0}^{i+j-m-1} \mathbb{E}\left[\sum_{k=1}^{R_{i,j-l,l}} Y_{i,j-l,l}^{(k)} \middle| \mathcal{N}_m, \mathcal{R}_m, \mathcal{X}_m \right].$$

Using assumptions (A1)–(A3) we obtain the following result.

Proposition 1

$$Z_m^{\text{RBNS}} = \sum_{i=2}^{m} \nu_i \sum_{j=m-i+1}^{m-1} \sum_{l=i+j-m}^{j} N_{i,j-l}\ \pi_l\ \mu_{j-l,l}, \tag{14}$$

$$Z_m^{\text{IBNR}} = \sum_{i=2}^{m} \vartheta_i\ \nu_i \sum_{j=m-i+1}^{m-1} \sum_{l=0}^{i+j-m-1} \beta_{j-l}\ \pi_l\ \nu_i\ \mu_{j-l,l}. \tag{15}$$

Using the previous expressions we can estimate the RBNS claims reserve by plugging estimates of the parameters in (14) and similarly the IBNR reserve using (15). Denote the resulting predictions by $\widehat{Z}_m^{\text{RBNS}}$ and $\widehat{Z}_m^{\text{IBNR}}$, respectively. Then the total reserve can be estimated by $\widehat{Z}_m = \widehat{Z}_m^{\text{RBNS}} + \widehat{Z}_m^{\text{IBNR}}$. A straightforward calculation demonstrates that the model defined in (A1)–(A3) can provide the same reserve as the classical chain ladder just by making a particular choice. This result is stated in the following corollary.

Corollary 1 *Under the additional assumptions that* $\widehat{\vartheta}_i^{(1)} = \widehat{\vartheta}_i^{(2)}$*, for all* $i = 2, \ldots, m$*, and* $N_{i,j} = \widehat{\vartheta}_i \widehat{\beta}_j$*, for all* $(i,j) \in \mathcal{I}_m$*, we have* $\widehat{Z}_m = \widehat{Z}_m^{CL}$*.*

Often claims development goes beyond the latest development period $m-1$, which has been observed at time m. Therefore, in practice, one needs to add a tail estimate to the claims reserves in order to also cover these additionally expected outstanding loss liability cash flows. The entire tail can be estimated under assumptions (A1)–(A3) if we additionally assume that $\beta_j = \pi_j = 0$ for $j = 1, \ldots, m-1$. In this particular case, we know that all claims are reported after development period $j = m-1$. Thus, we define the claims reserves including the tail by (re-arranging the summations)

$$\widehat{Z}_m^{\text{RBNS+}} = \sum_{i=1}^{m} \widehat{\nu}_i \sum_{j=0}^{m-i} N_{i,j} \sum_{l=m-(i+j)+1}^{m-1} \widehat{\pi}_l\ \widehat{\mu}_{j,l},$$

$$\widehat{Z}_m^{\text{IBNR+}} = \sum_{i=2}^{m} \widehat{\vartheta}_i\ \widehat{\nu}_i \sum_{j=m-i+1}^{m-1} \widehat{\beta}_j \sum_{l=0}^{m-1} \widehat{\pi}_l\ \widehat{\mu}_{j,l},$$

and the total reserves including the tail are defined by $\widehat{Z}_m^{+} = \widehat{Z}_m^{\text{RBNS+}} + \widehat{Z}_m^{\text{IBNR+}}$.

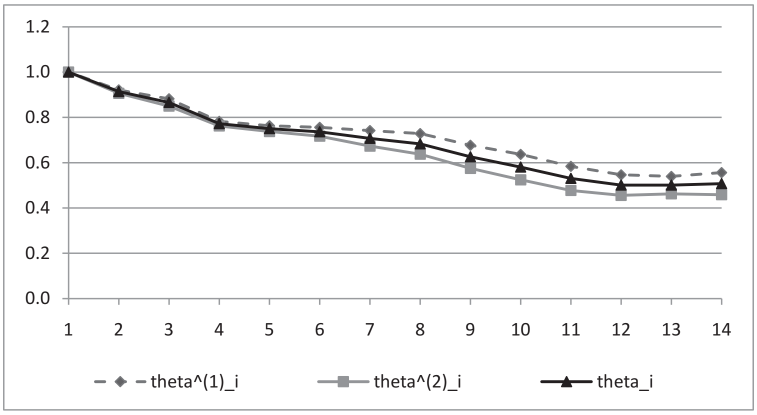

Figure 1: *Real data example: estimates for ϑ_i. Estimates $\widehat{\vartheta}_i^{(1)}$ are based on $\mathcal{N}_m$, estimates $\widehat{\vartheta}_i^{(2)}$ are based on $\mathcal{R}_m$ and $\widehat{\vartheta}_i$ is the arithmetic mean between the latter two estimates.*

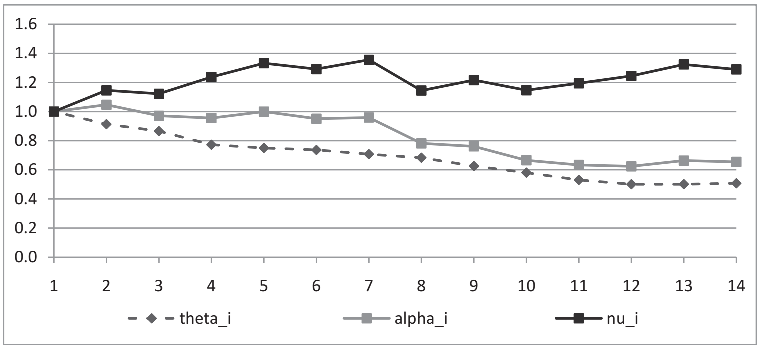

Figure 2: *Real data example: estimates for α_i, ϑ_i and ν_i.*

5. An example with real data

We illustrate the methods proposed in this paper using a real data example provided in Tables 6, 7 and 8 in Appendix C. The first step is to estimate the parameters according to Section 3.

In Figure 1 we give the estimates for ϑ_i for $i = 1, \ldots, m = 14$. We see that both data sets $\mathcal{N}_m$ and $\mathcal{R}_m$ provide similar estimates $\widehat{\vartheta}_i^{(1)}$ and $\widehat{\vartheta}_i^{(2)}$ for ϑ_i which confirms the model assumptions (A1)–(A3). Moreover, we see a strong decrease in the volume in this portfolio, since the exposure parameters $\widehat{\vartheta}_i$ decrease from 1 to roughly 0.5.

We could now proceed as described above and use the estimates $\widehat{\beta}_j$ and $\widehat{\lambda}_j$. However, we slightly deviate from this approach. Namely, if we plug in the resulting (adjusted) exposure estimates $\widehat{\vartheta}_i$ from (8) into (6) and (7) we get adjusted estimates $\widetilde{\beta}_j$ for β_j and similarly $\widetilde{\lambda}_j$ for λ_j. We prefer to work with these adjusted estimates because they assure that the overall level is correct if calculate the cross-classified means of $N_{i,j}$ and $R_{i,j}$, see Proposition 2.

In Figure 2 we show the estimates for the exposures α_i and ϑ_i, and the resulting inflation estimate $\widehat{\nu}_i$ is provided by the ratio of the latter two estimates. In general, we see an increase in the time-series $\widehat{\nu}_1, \ldots, \widehat{\nu}_{14}$, however accident year $i = 8$ seems conspicuous and needs further analysis on single claims data. It may indicate that there is a change in the underlying product (if it only acts on horizontal axis in the claims development triangle). Indeed we observe a substantial decrease in average payments per reported claim in accident year $i = 8$ which supports the argument of changes in the product (or portfolio).

Finally, we estimate π_l and $\mu_{j,l}$ from $\widetilde{\beta}_j$, $\widetilde{\lambda}_j$ and $\widehat{\gamma}_j$. We solve the estimation problem under assumption (10), i.e. $\mu_{j,l} \equiv \mu_l$. In that case we set $\widetilde{\mathbf{B}}_\beta = \mathbf{B}_{\widetilde{\beta}_0,\ldots,\widetilde{\beta}_{m-1}}$ and then we estimate π_l and $\pi_l \mu_l$ from equations (9) and (12). Figure 3 provides the estimates $\widehat{\pi}_l$. First of all we observe that all $\widehat{\pi}_l > 0$ except $\widehat{\pi}_2 < 0$ and $\widehat{\pi}_{12} < 0$ which contradicts the model assumptions (A1)–(A3). Thus, at this point we might ask for a more sophisticated model. However, this would also ask for more micro-level observations. We refrain from doing so but correct this value. In our particular case, we choose correction

$$\widetilde{\pi}_l = \begin{cases} \widehat{\pi}_l - 2|\widehat{\pi}_{l+1}| & l = 1, \\ |\widehat{\pi}_l| & l = 2, \\ 0 & l = 12, 13, \\ \widehat{\pi}_l & \text{otherwise.} \end{cases}$$

The resulting adjusted estimates $\widetilde{\pi}_l$ are also given in Figure 3. Note that we have $\sum_l \widetilde{\pi}_l \approx \sum_l \widehat{\pi}_l = 0.7251$, which says that on average we expect 0.7251 payments per reported claim, and in the average almost half of the claims can be settled without a payment. An analysis of payments per reported claim shows that this figure is decreasing

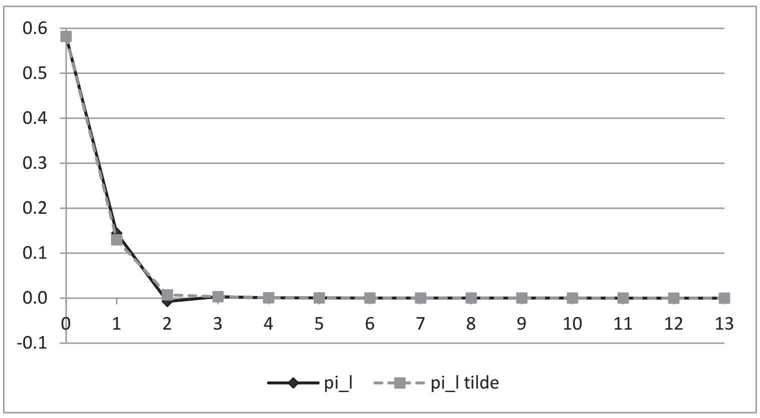

Figure 3: *Real data example: estimates $\widehat{\pi}_l$ and $\widetilde{\pi}_l$ for π_l.*

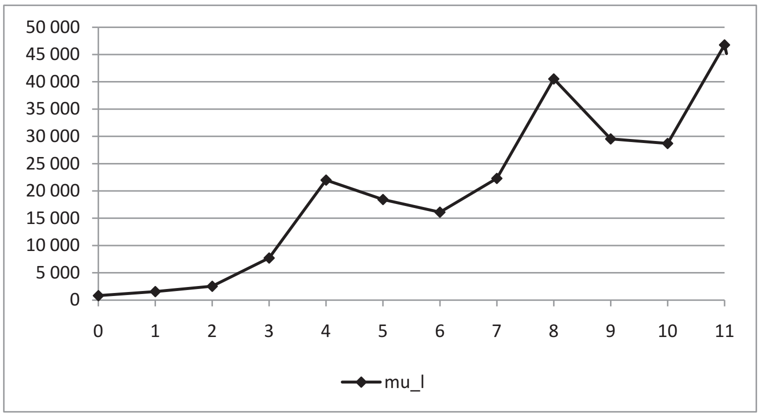

Figure 4: *Real data example: estimates $\widehat{\mu}_{j,l}$ for $\mu_{j,l} \equiv \mu_l$ for $l = 0, \ldots, 11$.*

over time. This decrease can have various reasons such as changes in reporting philosophy, changes in the claims handling process, but it could also be related to changes in the portfolio (we have already mentioned that the volume is strongly decreasing).

We then estimate $\mu_l \pi_l$ from $\widehat{\mu\pi}_l$, which is the solution of the system (12). And, under (10), we estimate $\mu_{j,l} = \mu_l$ by $\widehat{\mu\pi}_l / \widetilde{\pi}_l$.

The results are presented in Figure 4. We see that the average payments $\mu_{j,l}$ are increasing in the payment delay l. We could now further smooth this curve for the expected payments $\mu_{j,l}$, but we refrain from doing so. There are also other issues, for example that the payments $Y^{(k)}_{i,j,l}$ may not only depend on the accident year i and the payment delay l but also on the reporting delay j. However, as described in Section A we cannot model all directions simultaneously because this would lead to an over-parametrization.

Finally, in Table 1 we present the resulting claims reserves. We observe that under assumptions (A1)–(A3) and (10) we obtain higher claims reserves than classical chain ladder (see the last two columns in Table 1). One reason for this more conservative estimate is that we judge the upper right corner of the triangle $\mathcal{X}_m$ differently. The estimate for later development periods, say $j = 11, 12, 13$, is based on a rather small set observations in the chain ladder method (and hence not very reliable). In our model we use the additional model structure for the estimation of payments in later development periods which, in this case, is more conservative. The influence of the tail estimate is only minor, specifically $\widehat{Z}^+_m - \widehat{Z}_m = 7\,074$. This has to do with the fact that we have a rather short payout pattern $\widetilde{\pi}_l$ (see Figure 3).

Another possible approach in the previous calculations is to use condition (11), i.e. $\mu_{j,l} \equiv \mu_j$. However, the resulting claims reserves derived in this case seemed to be too low and we have decided not to include this in the paper. The reason is because the main driver of late payments is the payment delay π_l and not the reporting delay β_j. This implies that under (11) we underestimate the amounts of late payments because they are attached too strongly to the reporting pattern β_j compared to the payment pattern π_l.

Table 1: *Real data example: resulting claims reserves under (10).*

a.y. i	$\widehat{Z}_m^{\text{RBNS}+}$	$\widehat{Z}_m^{\text{IBNR}+}$	$\widehat{Z}_m^{+}$	$\widehat{Z}_m^{CL}$	difference	in %
1	536	0	536		536	
2	1 540	0	1 540	0	1 540	
3	23 799	0	23 799	2 220	21 579	971.8%
4	162 275	0	162 275	147 434	14 841	10.1%
5	291 122	790	291 912	280 056	11 855	4.2%
6	415 955	1 590	417 545	408 154	9 391	2.3%
7	584 991	3 300	588 291	569 060	19 231	3.4%
8	605 767	3 676	609 443	583 785	25 658	4.4%
9	704 687	5 039	709 726	675 363	34 363	5.1%
10	803 884	6 343	810 228	764 373	45 855	6.0%
11	1 054 124	10 037	1 064 161	1 004 331	59 829	6.0%
12	1 397 607	22 068	1 419 675	1 352 819	66 856	4.9%
13	1 999 243	84 680	2 083 922	2 076 674	7 248	0.3%
14	4 221 084	1 474 793	5 695 877	5 487 650	208 227	3.8%
total	12 266 615	1 612 315	13 878 930	13 351 921	527 009	3.9%

6. Bootstrap predictive distribution

6.1. Conditional mean square error of prediction

In addition to the claims reserves estimates $\widehat{Z}_m^{+}$ we also need to assess the corresponding prediction uncertainty. We briefly describe this with the help of the conditional mean square error of prediction (MSEP) uncertainty measure which is defined by

$$\text{msep}_{X_m|\{\mathcal{N}_m,\mathcal{R}_m,\mathcal{X}_m\}}\left(\widehat{Z}_m^{+}\right) = \mathbb{E}\left[\left(X_m - \widehat{Z}_m^{+}\right)^2 \middle| \mathcal{N}_m, \mathcal{R}_m, \mathcal{X}_m\right], \tag{16}$$

where the aggregate cash flow in the lower triangle is defined by $X_m = \sum_{i=2}^{m}\sum_{j=m-i+1}^{m-1} X_{i,j}$. Thus, the conditional MSEP describes the possible fluctuations of the true outstanding loss liability cash flows X_m around the predictor $\widehat{Z}_m^{+}$. Since the predictor is $\sigma\{\mathcal{N}_m, \mathcal{R}_m, \mathcal{X}_m\}$-measurable the conditional MSEP can be decoupled into process variance and parameter estimation error, see (3.1) in Wüthrich and Merz (2008),

$$\text{msep}_{X_m|\{\mathcal{N}_m,\mathcal{R}_m,\mathcal{X}_m\}}\left(\widehat{Z}_m^{+}\right) = \text{Var}\left(X_m | \mathcal{N}_m, \mathcal{R}_m, \mathcal{X}_m\right) + \left(Z_m - \widehat{Z}_m^{+}\right)^2. \tag{17}$$

The first term (process variance) can be calculated explicitly under our model assumptions, the second term (parameter estimation error) is more difficult to assess. Often, one derives approximations for this latter term. However, in our case this is too involved, therefore we rely on the bootstrap simulation method to quantify the prediction uncertainty. In order to apply the bootstrap method there is the parameter $s^2_{j,l}$ that still needs to be estimated.We do this under calibration (10), i.e. we set

$$s^2_{j,l} \equiv s^2_l \tag{18}$$

to avoid over-parameterization. In view of Proposition 5 we have

$$\mathbb{E}\left[\frac{X_{i,j} - \alpha_i\,\gamma_j}{\sqrt{\alpha_i\,\nu_i}}\right] = 0 \qquad \text{and} \qquad \mathrm{Var}\left(\frac{X_{i,j} - \alpha_i\,\gamma_j}{\sqrt{\alpha_i\,\nu_i}}\right) = \sigma^2_j.$$

The sample estimator then provides estimates

$$\widehat{\sigma}^2_j = \frac{1}{m-j-1}\sum_{i=1}^{m-j}\left(\frac{X_{i,j} - \widehat{\alpha}_i\,\widehat{\gamma}_j}{\sqrt{\widehat{\alpha}_i\,\widehat{\nu}_i}}\right),$$

for $j = 0,\ldots,m-2$ and we set $\widehat{\sigma}^2_{m-1} = \widehat{\sigma}^2_{m-2}$. In view of (21) we have a second description for σ^2_j. If we solve this for s^2_l and replace all parameters by their estimates we obtain estimates

$$((\widehat{\pi s^2})_0,\ldots,(\widehat{\pi s^2})_{m-1})^\top = \widetilde{B}_\beta^{-1}(\widehat{\sigma}^2_0,\ldots,\widehat{\sigma}^2_{m-1})^\top - (\widetilde{\pi}^2_0\,\widehat{\mu}^2_0,\ldots,\widetilde{\pi}^2_{m-1}\,\widehat{\mu}^2_{m-1})^\top,$$

and finally we set

$$\widehat{s}^2_l = (\widehat{\pi s^2})_l/\widetilde{\pi}_l, \qquad \text{for all } l = 0,\ldots,m-1. \tag{19}$$

If we apply this procedure to Example 1 we obtain the result in Table 2. In order to justify these estimates we calculate the estimates of the corresponding coefficients of variation given by $\widehat{\mathrm{vco}} = \widehat{s}_l/\widehat{\mu}_l$. Table 2 shows that these estimated coefficients of variation are in the interval $[1.5, 5.5]$, i.e. the coefficients of variation for single claims payouts $Y^{(k)}_{i,j,l}$ are of order 1.5 to 5.5. These are reasonable values, for instance in the Swiss Solvency Test (SST) the coefficients of variation for single claim sizes (not payouts) are estimated between 2.25 and 11 depending on the underlying line of business, see Section 8.4.4 in FINMA (2006). These estimators now allow for applying bootstrap methods which are

Table 2: *Real data example: resulting standard deviation estimates $\widehat{s}_l$ together with the mean estimates $\widehat{\mu}_l$ and the corresponding coefficient of variation estimates $\widehat{\mathrm{vco}}$.*

	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{s}_l$	2 862	8 511	11 651	26 688	93 291	28 083	52 846	43 333	104 714	59 276	75 632	104 701
$\widehat{\mu}_l$	818	1 561	2 534	7 712	21 993	18 435	16 113	22 300	40 529	29 540	28 704	46 764
$\widehat{\mathrm{vco}}$	3.50	5.45	4.60	3.46	4.24	1.52	3.28	1.94	2.58	2.01	2.63	2.24

Table 3: *Real data example: process standard deviation (first row) and rooted conditional MSEP (second row) under model (A1)-(A3) for the predicted RBNS, IBNR and the total claim reserves. The last column gives the results of the Mack formula (Mack 2010).*

	RBNS	IBNR	total	Mack (1993)
process standard deviation	1 511 860	293 166	1 545 503	1 521 713
conditional MSEP$^{1/2}$	2 273 294	326 382	2 324 966	2 182 722

close to those proposed by Martínez-Miranda *et al.* (2011, 2012). Specifically, we derive the predictive distribution using a parametric bootstrapping procedure which exploits the model assumptions in Section 2. In a first step we define a bootstrapping scheme based on Monte Carlo simulation from the model (A1)–(A3) where the unknown parameters are simply replaced by the estimated parameters (ignoring the parameter estimation uncertainty). This gives an estimation of the process variance defined as the first term in equation (17). The resulting process variances for RBNS, IBNR and total reserves (for all the years) are given in the first row of Table 3. To quantify the second term in equation (17), i.e. the parameter estimation error, we consider a more general bootstrap algorithm which also simulates the distribution of the involved parameters. From such general bootstrap method – formally described below – we derive the desired conditional MSEP. The resulting errors are displayed in the second row of Table 3. The last column displays the same uncertainties obtained from the Mack's distribution-free chain ladder model Mack (1993). We observe that our bootstrap results are slightly more conservative compared to the classical Mack formula.

6.2. Bootstrapping the RBNS and IBNR reserve

The predictive distribution which describes the possible fluctuations of the true outstanding loss liability cash flows can be derived using parametric bootstrap methods. By exploiting the distributional assumptions (A1)–(A3) we describe in the Appendix B an explicit algorithm to derive separately the predictive distribution of the RBNS and IBNR cash flows, X_m^{RBNS} and X_m^{IBNR}, respectively. With this resampling scheme the RBNS and IBNR cash flows can be simulated using Monte Carlo methods. We have derived these cash flows for the data in Example 1. Table 4 shows the median and the upper quantiles separately for the RBNS and IBNR cash flows. Here we consider $B = 10\,000$ replications in the resampling scheme. As we expect the means imitate the predicted reserves given in Table 1. The calculated medians however are slightly lower, which reveals that the derived distribution is negatively skewed.

For comparison purposes we also consider the double chain ladder method (DCL) proposed by Martínez-Miranda *et al.* (2012). This method is defined under a simpler distributional model which makes the following assumptions on the first two moments $\mathbb{E}[Y_{i,j,l}^{(1)}] = \nu_i \mu$ and $\mathbb{E}[(Y_{i,j,l}^{(1)})^2] = \nu_i^2 \sigma^2$. Table 5 reports the summary of the distribution for

Table 4: *Real data example. Simulation of predictive distribution of RBNS and IBNR reserves by accident year: mean, median and* 95% *and* 99% *quantiles over* $B = 10\,000$ *repetitions. Column 2–5 give the RBNS reserves, Column 6–9 give the IBNR reserves.*

	RBNS				IBNR			
a.y. i	mean	median	95%	99%	mean	median	95%	99%
1	522	0	0	882	0	0	0	0
2	1658	0	0	38 893	0	0	0	0
3	23 947	0	140 730	352 637	0	0	0	0
4	165 490	73 036	633 033	1 172 022	0	0	0	0
5	297 554	199 030	932 461	1 579 736	202	0	18	3 910
6	418 854	321 105	1 106 734	1 840 653	688	0	2 509	16 514
7	586 159	476 807	1 435 075	2 194 285	1 617	0	8 521	30 389
8	609 403	522 311	1 377 117	2 047 477	2 312	8	12 839	34 068
9	712 294	615 028	1 548 731	2 236 150	3 750	92	18 937	49 613
10	809 344	716 227	1 660 795	2 374 073	5 108	639	22 475	58 796
11	1 056 515	953 340	2 092 864	2 990 605	9 096	2 896	37 051	78 573
12	1 410 137	1 295 048	2 537 813	3 437 648	21 487	13 153	69 200	125 271
13	2 008 886	1 899 042	3 271 259	4 179 189	86 354	72 050	188 811	327 241
14	4 211 291	4 126 027	5 463 231	6 402 499	1 552 438	1 512 135	2 074 502	2 514 402
total	12 312 055	12 040 963	16 325 473	18 860 539	1 683 054	1 640 097	2 222 831	2 709 200

the RBNS, IBNR and total claims reserves. The resulting reserves are similar when we consider the sum over all accident years. However, we observe more variability in the method proposed in this paper, under assumptions (A1)–(A2), compared to the DCL method. This is due to the fact that in DCL method there is the assumption that a claim is settled by a single payment and hence there is less volatility in the cash flow process. Besides, the model in this paper involves more parameters than the DCL model and therefore it increases the uncertainty of the parameters, which we are taking into account in the resampling scheme (see algorithm in Appendix B).

Table 5: *Real data example. Bootstrap predictive distribution: RBNS, IBNR and total claims reserves. The first three columns give the summary of the distribution under model (A1)–(A3). The last three columns provide the bootstrap distribution from the DCL method proposed in Martínez-Miranda* et al. *(2012).*

	model (A1)-(A3)			DCL		
	RBNS	IBNR	total	RBNS	IBNR	total
mean	12 312 055	1 683 054	13 995 109	11 758 152	1 585 151	13 343 303
$\text{MSEP}^{1/2}$	2 273 294	326 382	2 324 966	1 881 154	485 312	2 018 112
1%	8 090 717	1 131 376	9 615 040	8 081 739	687 623	9 314 398
5%	9 088 207	1 262 754	10 685 634	9 012 040	897 886	10 408 658
50%	12 040 963	1 640 097	13 723 567	11 637 796	1 532 079	13 243 493
95%	16 325 473	2 222 831	18 101 695	14 869 197	2 448 915	16 729 435
99%	18 860 539	2 709 200	20 660 941	16 516 558	2 941 469	18 487 830

7. Conclusions

In this paper we have defined the claims reserving model on an individual claims processes basis (micro-level). The definition of the model on this micro-level has been done such that on the aggregate level we re-discover the classical chain ladder reserving method. Under this model we show how extended data collection can provide us with more and better information to act in time on unforeseen patterns of outstanding liabilities. In particular we have focused on how various claims delays impact severities and how to incorporate this information in the reserve. Our approach in this paper shares the simplicity and intuitive appeal which have popularized the chain ladder method in claims reserving. But, with a little more effort in calculations and data requirements, our approach reports several other advantages. Since chain ladder is only based in the aggregated payments triangles, it cannot provide the split of the claims reserves into RBNS and IBNR and the tail as we do. Such split is required for the calculation of unallocated loss adjustment expenses ULAE and it gives valuable information to the insurer. In addition, to work under a well-defined and firm statistical model provides a suitable framework to develop consistent bootstrap methods to quantify the uncertainty in the predictions. In future work we will also consider simulation of coefficients of variation following the insights of for example Gulhar, Kibria, Albatineh and Ahmed (2012).

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A. Moments calculations

Here we provide calculations about the two first moments of the stochastic variables in the triangles $\mathcal{N}_m$, $\mathcal{R}_m$ and $\mathcal{X}_m$. Hereafter we work under the model assumptions (A1)–(A3) formulated in Section 2.2.

A.1. Calculation of means

We start with the claims payments $X_{i,j,l}$ given in (A3). The conditional and unconditional means are given by

$$\mathbb{E}[X_{i,j,l}|N_{i,0},\dots,N_{i,m-1}] = N_{i,j}\ \pi_l\ \nu_i\ \mu_{j,l},$$
$$\mathbb{E}[X_{i,j,l}] = \mathbb{E}[\mathbb{E}[X_{i,j,l}|N_{i,0},\dots,N_{i,m-1}]] = \vartheta_i\ \beta_j\ \pi_l\ \nu_i\ \mu_{j,l}.$$

The total number of payments $R_{i,j}$ of accident year i in accounting year $i+j$ has, conditionally given $\{N_{i,0},\dots,N_{i,m-1}\}$, a Poisson distribution with conditional mean

$$\mathbb{E}[R_{i,j}|N_{i,0},\dots,N_{i,m-1}] = \sum_{l=0}^{j}\mathbb{E}[R_{i,j-l,l}|N_{i,j-l}] = \sum_{l=0}^{j} N_{i,j-l}\ \pi_l.$$

This implies for the unconditional mean

$$\mathbb{E}[R_{i,j}] = \mathbb{E}[\mathbb{E}[R_{i,j}|N_{i,0},\dots,N_{i,m-1}]] = \vartheta_i \sum_{l=0}^{j}\beta_{j-l}\ \pi_l.$$

Define $\lambda_j = \sum_{l=0}^{j}\beta_{j-l}\ \pi_l$, for $j = 0,\dots,m-1$, then we have just proved the following proposition.

Proposition 2 $\mathbb{E}[N_{i,j}] = \vartheta_i\ \beta_j$ *and* $\mathbb{E}[R_{i,j}] = \vartheta_i\ \lambda_j.$

Thus, the pair $(N_{i,j}, R_{i,j})$ satisfies the double chain ladder model of Martínez-Miranda *et al.* (2012) with inflation parameter set equal to 1. ϑ_i describes an exposure measure for accident year i, $(\beta_j)_j$ gives the claims reporting pattern and $(\lambda_j)_j$ provides the number of payment count pattern.

The accounting year payments $X_{i,j}$ for accident year i in accounting year $i+j$ have, conditionally given $\{N_{i,0},\dots,N_{i,m-1}\}$, a compound Poisson distribution with conditional mean

$$\mathbb{E}[X_{i,j}|N_{i,0},\dots,N_{i,m-1}] = \sum_{l=0}^{j} N_{i,j-l}\ \pi_l\ \nu_i\ \mu_{j-l,l}.$$

This provides the unconditional mean for $X_{i,j}$ given by

$$\mathbb{E}[X_{i,j}] = \vartheta_i \; \nu_i \; \sum_{l=0}^{j} \beta_{j-l} \; \pi_l \; \mu_{j-l,l}.$$

We define the parameter γ_j which only depends on the development period j given by $\gamma_j = \sum_{l=0}^{j} \beta_{j-l} \; \pi_l \; \mu_{j-l,l}$. Thus, we obtain a cross-classified unconditional first moment for $X_{i,j}$ which is stated in the following proposition.

Proposition 3 *We have for* $\alpha_i = \vartheta_i \; \nu_i$ *that* $\mathbb{E}[X_{i,j}] \; = \; \alpha_i \; \gamma_j$.

This moment property is similar to the Bornhuetter-Ferguson models used by Mack (2008) and Saluz, Gisler and Wüthrich (2011), Models 4.11 and 4.16. Moreover, Proposition 3 explains how the claims development reporting pattern $(\beta_j)_j$ for $N_{i,j}$ is related to the claims development pattern $(\gamma_j)_j$ for claims payments $X_{i,j}$.

A.2. Calculation of variances

In a similar fashion to the first moments we calculate the variances. First we have under the conditional compound Poisson assumptions (A3)

$$\mathrm{Var}\left(X_{i,j,l} | N_{i,0}, \ldots, N_{i,m-1}\right) \; = \; N_{i,j} \; \pi_l \; \nu_i^2 \; s_{j,l}^2,$$

and for the unconditional variance we have

$$\begin{aligned}\mathrm{Var}\left(X_{i,j,l}\right) &= \mathrm{Var}\left(\mathbb{E}\left[X_{i,j,l} | N_{i,0}, \ldots, N_{i,m-1}\right]\right) + \; \mathbb{E}\left[\mathrm{Var}\left(X_{i,j,l} | N_{i,0}, \ldots, N_{i,m-1}\right)\right] \\ &= \vartheta_i \; \beta_j \; \nu_i^2 \left(\pi_l^2 \; \mu_{j,l}^2 + \pi_l \; s_{j,l}^2\right).\end{aligned}$$

The total number of payments $R_{i,j}$ of accident year i in accounting year $i+j$ has, conditionally given $\{N_{i,0}, \ldots, N_{i,m-1}\}$, a Poisson distribution with conditional variance

$$\mathrm{Var}\left(R_{i,j} | N_{i,0}, \ldots, N_{i,m-1}\right) = \sum_{l=0}^{j} \mathrm{Var}\left(R_{i,j-l,l} | N_{i,j-l}\right) = \sum_{l=0}^{j} N_{i,j-l} \; \pi_l.$$

This implies for the unconditional variance

$$\begin{aligned}\mathrm{Var}\left(R_{i,j}\right) &= \mathrm{Var}\left(\mathbb{E}\left[R_{i,j} | N_{i,0}, \ldots, N_{i,m-1}\right]\right) + \mathbb{E}\left[\mathrm{Var}\left(R_{i,j} | N_{i,0}, \ldots, N_{i,m-1}\right)\right] \\ &= \; \vartheta_i \; \sum_{l=0}^{j} \beta_{j-l} \; \pi_l^2 + \vartheta_i \sum_{l=0}^{j} \beta_{j-l} \; \pi_l.\end{aligned}$$

Define for $j = 0, \ldots, m-1$

$$t_j^2 = \sum_{l=0}^{j} \beta_{j-l}\ \pi_l\,(1+\pi_l) \geq \lambda_j, \tag{20}$$

then we have just proved the following proposition.

Proposition 4 $\operatorname{Var}(N_{i,j}) = \vartheta_i\,\beta_j$ *and* $\operatorname{Var}(R_{i,j}) = \vartheta_i\,t_j^2$.

In view of Proposition 2 we see that for the number of payments $R_{i,j}$ we obtain over-dispersion parameter

$$\phi_j = \frac{t_j^2}{\lambda_j} = 1 + \frac{\sum_{l=0}^{j} \beta_{j-l}\ \pi_l^2}{\sum_{l=0}^{j} \beta_{j-l}\ \pi_l} \geq 1.$$

Note that $R_{i,j}$ has a mixed Poisson distribution which is exactly reflected in this over-dispersion parameter $\phi_j \geq 1$.

The accounting year payments $X_{i,j}$ for accident year i in accounting year $i+j$ have, conditionally given $\{N_{i,0}, \ldots, N_{i,m-1}\}$, a compound Poisson distribution with conditional variance

$$\operatorname{Var}(X_{i,j} | N_{i,0}, \ldots, N_{i,m-1}) = \sum_{l=0}^{j} N_{i,j-l}\ \pi_l\ \nu_i^2\ s_{j-l,l}^2.$$

This provides the unconditional variances for $X_{i,j}$ given by

$$\begin{aligned}\operatorname{Var}(X_{i,j}) &= \operatorname{Var}(\mathbb{E}[X_{i,j}|N_{i,0}, \ldots, N_{i,m-1}]) + \mathbb{E}[\operatorname{Var}(X_{i,j}|N_{i,0}, \ldots, N_{i,m-1})]\\ &= \vartheta_i\ \nu_i^2\ \sum_{l=0}^{j} \beta_{j-l}\ \pi_l^2\ \mu_{j-l,l}^2 + \vartheta_i\ \nu_i^2\ \sum_{l=0}^{j} \beta_{j-l}\ \pi_l\ s_{j-l,l}^2.\end{aligned}$$

We define the parameter σ_j^2 which only depends on the development period j given by

$$\sigma_j^2 = \sum_{l=0}^{j} \beta_{j-l}\ \pi_l\ \mu_{j-l,l} \left(\pi_l\ \mu_{j-l,l} + \frac{s_{j-l,l}^2}{\mu_{j-l,l}} \right). \tag{21}$$

Thus, we obtain a cross-classified model for $X_{i,j}$ with first moment given by $\mathbb{E}[X_{i,j}] = \alpha_i\ \gamma_j$ and variance given in the following proposition:

Proposition 5 $\operatorname{Var}(X_{i,j}) = \alpha_i\ \nu_i\ \sigma_j^2$.

Again it is similar to the claims reserving models used in Mack (2008) and Saluz *et al.* (2011), Models 4.11 and 4.16, but now the parameters have an explicit meaning.

B. Resampling schemes

Here we provide the algorithm to derive the predictive distribution of the RBNS and IBNR cash flow: X_m^{RBNS} and X_m^{IBNR}. We denote by $\theta = \{\pi_l, \mu_l, s_l, \nu_i;\ l = 0, \ldots, m-1, i = 1, \ldots, m\}$ the set of parameters involved in the model, under calibration (10). Moreover, let $\widehat{\theta}$ denote the parameters estimated from the data $(\mathcal{N}_m, \mathcal{R}_m, \mathcal{X}_m)$ which can be calculated using the methods described in Section 3 and expression (19).

Algorithm RBNS

Step 1. *Estimation of the parameters and distributions.* From the observed data $(\mathcal{N}_m, \mathcal{R}_m, \mathcal{X}_m)$ estimate the model parameters θ by the estimator $\widehat{\theta} = \{\widehat{\pi}_l, \widehat{\mu}_l, \widehat{s}_l, \widehat{\nu}_i;\ l = 0, \ldots, m-1, i = 1, \ldots, m\}$, as described above. The payment delay distribution is estimated by a Poisson distribution with estimated parameter, i.e. $R_{i,j,l}|_{\{N_{i,0},\ldots,N_{i,m-1}\}} \sim \mathrm{Poi}(N_{i,j}\widehat{\pi}_l)$. The distribution of the individual payments, $Y_{i,j,l}^{(1)}$ is estimated by a gamma distribution with shape parameter $\widehat{\lambda} = \widehat{\mu}_l^2/(\widehat{s}_l^2 - \widehat{\mu}_l^2)$ and scale parameter $\widehat{\kappa} = (\widehat{s}_l^2 - \widehat{\mu}_l^2)\widehat{\nu}_i/\widehat{\mu}_l$.

Step 2. *Bootstrapping the data*. Conditional on the observed number of reported claims $\mathcal{N}_m$ generate new bootstrapped triangles $\mathcal{R}_m^* = \{R_{i,j}^*;\ i+j \leq m\}$ and $\mathcal{X}_m^* = \{X_{i,j}^*;\ i+j \leq m\}$ as follows:

(i) Simulate the payment delay: from each $N_{i,j}$, $i+j \leq m$, generate the number of payments, $R_{i,j,l}^*$ from a Poisson distribution with parameter $N_{i,j}\widehat{\pi}_l$ estimated in Step 1. Calculate the bootstrapped total number of payments, $\mathcal{R}_m^* = \{R_{i,j}^*;\ i+j \leq m\}$ from expression (1).

(ii) Get the bootstrapped aggregated payments, $\mathcal{X}_m^* = \{X_{i,j}^*;\ i+j \leq m\}$, from the gamma distribution estimated in Step 1 and using expression (2) but replace $R_{i,j-l,l}$ by $R_{i,j-l,l}^*$.

Step 3. *Bootstrapping the parameters.* From the bootstrap data, $(\mathcal{R}_m^*, \mathcal{X}_m^*)$, and the original $\mathcal{N}_m$, estimate again the parameters and get bootstrapped parameters θ^*.

Step 4. *Bootstrapping the RBNS predictions.* Simulate the RBNS cash flow, $X_m^{\mathrm{RBNS}*}$, in the lower triangle using similar specifications to (i) and (ii) in Step 2 but with bootstrapped parameters θ^*.

Step 5. *Monte Carlo approximation*. Repeat Steps 2-4 B times and get the empirical bootstrap distribution of the RBNS cash flows $\{X_m^{\mathrm{RBNS},b}; b = 1, \ldots, B\}$.

The IBNR algorithm to simulate the IBNR cash flows $X_m^{\mathrm{IBNR}*}$ follows the same steps as the algorithm RBNS but, in addition, involves the estimation and the simulation of the number of reported claims $N_{i,j}$ in the lower triangle $\mathcal{J}_m$. In this case and under assumption (A1), we simulate $\mathcal{N}_m^* = \{N_{i,j}^*;\ (i,j) \in \mathcal{I}_m\}$ from a Poisson distribution with parameters estimated by the chain ladder estimates $\{\widehat{\vartheta}_i, \widehat{\beta}_j; i, j+1 = 1, \ldots, m\}$ (for more details we also refer to Martínez-Miranda *et al.* 2011).

C. Run-off triangles

Table 6: *Example 1, number of reported claims $N_{i,j}$, $(i,j) \in \mathfrak{I}_m$.*

ay / dy	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	18 247	3 083	124	22	5	5	3	1	0	1	1	0	0	0
2	17 098	2 567	98	25	6	1	1	3	0	1	0	0	0	
3	16 110	2 700	107	18	7	5	4	1	4	0	0	0		
4	14 426	2 253	103	17	10	3	2	1	1	1	0			
5	14 142	2 173	62	11	7	4	0	1	1	0				
6	14 275	1 850	86	25	6	2	0	0	1					
7	14 019	1 797	97	19	5	1	1	1						
8	13 933	1 602	84	24	6	3	1							
9	12 962	1 503	65	11	2	2								
10	12 226	1 352	74	18	7									
11	11 124	1 347	57	12										
12	10 360	1 307	56											
13	10 371	1 141												
14	10 435													

Table 7: *Example 1, number of payments $R_{i,j}$, $(i,j) \in \mathfrak{I}_m$.*

ay / dy	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	11 761	4 800	324	71	39	14	10	6	3	5	5	2	2	0
2	10 927	4 077	303	60	28	12	13	5	8	4	5	5	0	
3	9 856	4 168	294	71	23	23	16	10	9	4	4	3		
4	8 915	3 682	246	70	27	16	7	7	4	7	4			
5	8 854	3 340	265	46	33	9	4	6	2	5				
6	8 881	3 000	199	70	22	15	8	8	4					
7	8 170	2 983	221	46	18	8	5	6						
8	7 827	2 741	184	55	22	15	3							
9	6 999	2 540	166	44	18	7								
10	6 240	2 420	184	45	18									
11	5 652	2 210	184	45										
12	5 223	2 317	148											
13	5 627	2 024												
14	5 483													

Table 8: *Example 1, claims payments $X_{i,j}$, $(i,j) \in \mathfrak{I}_m$.*

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	9 829 717	5 690 608	874 882	420 112	154 884	55 497	46 239	313 960	290 204	12 936	6 218	18 755	4 678	0
2	9 263 718	5 004 173	971 523	660 324	208 000	531 391	495 368	48 367	566 099	49 905	362 747	388 190	0	
3	9 402 126	5 625 116	805 027	322 263	325 505	101 469	160 747	310 837	30 754	69 395	8 123	51 756		
4	8 650 875	5 150 702	752 354	802 485	209 590	466 859	197 654	41 763	25 349	367 750	123 091			
5	8 848 118	4 748 516	1 390 699	1 140 610	412 090	359 991	20 169	220 227	54 395	240 967				
6	9 070 691	5 890 678	519 808	539 202	127 701	86 472	122 060	83 853	6 660					
7	8 763 254	4 293 444	1 339 396	292 330	1 515 615	155 402	28 210	36 709						
8	7 777 082	4 145 234	642 816	504 127	92 030	101 250	6 620							
9	7 212 984	3 498 230	778 132	354 855	626 442	342 182								
10	6 265 457	3 737 631	546 644	182 490	297 995									
11	5 737 447	3 281 469	748 102	456 983										
12	5 612 232	3 495 586	593 774											
13	6 386 024	3 289 703												
14	6 110 750													