QÜESTIIÓ, vol. 24, 3, p. 419-424, 2000

# A NOTE ON THE MATRIX HAFFIAN

HEINZ NEUDECKER
Cesaro*

*This note contains a transparent presentation of the matrix Haffian. A basic theorem links this matrix and the differential of the matrix function under investigation, viz* $\nabla F(X)$ *and* $dF(X)$.

*Frequent use is being made of matrix derivatives as developed by Magnus and Neudecker.*

**Keywords:** Matrix differentials, matrix functions

**AMS Classification (MSC 2000):** primary 62F10, secondary 62C99

* Oosterstraat, 13. 1741 GH Schagen. The Netherlands.
E-mail: heinz@fee.uva.nl

– Received February 2000.
– Accepted September 2000.

## 1. INTRODUCTION

Haff (1981, 1982) introduced a matrix function based on the derivatives of the elements of a **square** matrix function $F(X)$ of a **symmetric** matrix argument $X$.

We shall name it «the matrix Haffian». It was used by Haff in various multivariate statistical problems.

In this note we shall attempt to give a transparent presentation of the matrix Haffian, and give some applications.

Basic is a differentiable square matrix function $F(X)$, shortly $F$, which depends on a **symmetric** matrix $X$. Both matrices have the same dimension. A strategic rôle is being played by a square matrix $\nabla = (d_{ij})$ of differential operators $d_{ij} := 1/2(1+\delta_{ij})\dfrac{\partial}{\partial x_{ij}}$, where $\delta_{ij}$ is the Kronecker delta ($\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$). Haff used the symbol $D$, and not $\nabla$. In earlier work on the scalar Haffian (Neudecker 2000) the symbol $\nabla$ was introduced. This was done to avoid confusion with the duplication matrix which was extensively used then. The matrix $\nabla$ will be applied to $F$ and produce $\nabla F$, the matrix Haffian.

In the article frequent use will be made of matrix differentials and derivatives as proposed by Magnus and Neudecker (1999).

## 2. THE MATRIX HAFFIAN

Consider a differentiable square matrix function $F(X)$ with **symmetric** matrix argument $X$, both of dimension $m$. The application of $\nabla = (d_{ij})$, a (square) matrix of differential operators $d_{ij} := 1/2(1+\delta_{ij})\dfrac{\partial}{\partial x_{ij}}$ to $F$ yields $\nabla F$. Its $ik^{th}$ typical element is

$$\Sigma_j d_{ij} f_{jk} = 1/2\Sigma_j(1+\delta_{ij})\frac{\partial f_{jk}}{\partial x_{ij}} = \frac{\partial f_{ik}}{\partial x_{ii}} + 1/2\sum_{i\neq j}\frac{\partial f_{jk}}{\partial x_{ij}}.$$

We shall prove a basic result.

***Theorem***

*When $dF = P(dX)Q'$, then $\nabla F = 1/2P'Q' + 1/2(trP)Q'$.*

*Proof*

Using

$$dX = \Sigma_{ij}(dx_{ij})E_{ij},$$

where

$$X = \Sigma_{ij}x_{ij}E_{ij}$$

and $E_{ij}$ is the $ij^{th}$ basis matrix, we get

$$\begin{aligned} dF &= P(dX)Q' = \Sigma_{ij}(dx_{ij})PE_{ij}Q' = \\ &= \Sigma_i(dx_{ii})PE_{ii}Q' + \sum_{i\neq j}(dx_{ij})PE_{ij}Q'. \quad (i,j = 1\ldots m) \end{aligned}$$

Hence

(1)
$$\frac{\partial F}{\partial x_{ii}} = PE_{ii}Q'$$

(2)
$$\frac{\partial F}{\partial x_{ij}} = P(E_{ij}+E_{ji})Q'. \quad (i \neq j)$$

The second expression follows from the symmetry of $X$, where $x_{ij} = x_{ji} \quad (i \neq j)$.

Consider the $ik^{th}$ typical element of $\nabla F$, viz

$$\begin{aligned} \Sigma_j d_{ij} f_{jk} &= \frac{\partial f_{ik}}{\partial x_{ii}} + 1/2\sum_{j\neq i}\frac{\partial f_{jk}}{\partial x_{ij}} \\ &= e_i'\frac{\partial F}{\partial x_{ii}}e_k + 1/2\sum_{j\neq i}e_j'\frac{\partial F}{\partial x_{ij}}e_k \\ &= e_i'PE_{ii}Q'e_k + 1/2\sum_{j\neq i}e_j'P(E_{ij}+E_{ji})Q'e_k \\ &= e_i'Pe_ie_i'Q'e_k + 1/2\sum_{j\neq i}e_j'P(e_ie_j'+e_je_i')Q'e_k \\ &= p_{ii}q_{ki} + 1/2\sum_{j\neq i}(p_{ji}q_{kj}+p_{jj}q_{ki}) \\ &= 1/2\Sigma_j p_{ji}q_{kj} + 1/2\Sigma_j p_{jj}q_{ki} \\ &= 1/2(QP)_{ki} + 1/2(\mathrm{tr}P)q_{ki} \\ &= 1/2(P'Q')_{ik} + 1/2(\mathrm{tr}P)(Q')_{ik}. \end{aligned}$$

Hence $(\nabla F)_{ik} = 1/2[P'Q' + (\mathrm{tr}P)Q']_{ik}$,

$$\nabla F = 1/2P'Q' + 1/2(\mathrm{tr}P)Q'.$$

We used $E_{ij} = e_i e_j'$, where $e_i$ and $e_j$ are basis vectors. Further $(A)_{ik} = a_{ik} = e_i' A e_k$ was applied.

□

## 3. SOME MATRIX HAFFIANS

We consider four matrix Haffians. They are of a simple nature. For one we quote the literature.

(1) $$\nabla PX^{-1}Q' = -1/2X^{-1}P'X^{-1}Q' - 1/2(\mathrm{tr}PX^{-1})X^{-1}Q'.$$

*Proof*

Now

$$dF = dPX^{-1}Q' = P(dX^{-1})Q' = -PX^{-1}(dX)X^{-1}Q',$$

by virtue of $dX^{-1} = -X^{-1}(dX)X^{-1}$.

Using the Theorem with the substitutions $P \longrightarrow -PX^{-1}$ and $Q' \longrightarrow X^{-1}Q'$ we get the result immediately.

□

**Note.** Haff (1982, Lemma 6i) gave $\nabla QX^{-1}$ by using a long-winded procedure. We have $\nabla QX^{-1} = -1/2X^{-1}Q'X^{-1} - 1/2(\mathrm{tr}QX^{-1})X^{-1}$.

(2) $$\nabla PXQXR' = 1/2P'QXR' + 1/2(\mathrm{tr}P)QXR' + 1/2Q'XP'R' + 1/2(trPXQ)R'.$$

*Proof*

$$dPXQXR' = P(dX)QXR' + PXQ(dX)R',$$

which gives the substitutions

$$P \longrightarrow P, \quad Q' \longrightarrow QXR' \quad \text{and} \quad P \longrightarrow PXQ, \quad Q' \longrightarrow R'.$$

The Theorem completes the story.

□

(3) $$\nabla PX^{-2}Q' = -1/2X^{-1}P'X^{-2}Q' - 1/2(\mathrm{tr}PX^{-1})X^{-2}Q' - 1/2X^{-2}P'X^{-1}Q' - \\ - 1/2(\mathrm{tr}PX^{-2})X^{-1}Q'.$$

*Proof*

We have

$$\begin{aligned} dPX^{-2}Q' &= P(dX^{-2})Q' = P(dX^{-1})X^{-1}Q' + PX^{-1}(dX^{-1})Q' \\ &= -PX^{-1}(dX)X^{-2}Q' - PX^{-2}(dX)X^{-2}Q'. \end{aligned}$$

Application of the Theorem with the substitutions $P \longrightarrow -PX^{-1}$, $Q' \longrightarrow X^{-2}Q'$ and $P \longrightarrow -PX^{-2}$, $Q' \longrightarrow X^{-1}Q'$ finishes the derivation.

□

(4) $$\begin{aligned} \nabla PX^3Q' = {}& 1/2P'X^2Q' + 1/2(\mathrm{tr}P)X^2Q' + 1/2XP'XQ' + \\ & + 1/2(\mathrm{tr}PX)XQ' + 1/2X^2P'Q' + 1/2(\mathrm{tr}PX^2)Q'. \end{aligned}$$

*Proof*

$$dPX^3Q' = P(dX^3)Q = P(dX)X^2Q' + PX(dX)XQ' + PX^2(dX)Q'.$$

There are three sets of substitutions, viz.

$P \longrightarrow P, Q' \longrightarrow X^2Q'; P \longrightarrow PX, Q' \longrightarrow XQ'$ and $P \longrightarrow PX, Q' \longrightarrow Q'$.

This yields the result.

□

## 4. SOME REMARKS ON SCALAR HAFFIANS

The scalar Haffian is defined as $\mathrm{tr}\nabla F$.

See Neudecker (2000) for a thorough discussion and some applications. In that article the identity

$$\mathrm{tr}\nabla F = \mathrm{tr}\frac{\partial g}{\partial x'}$$

was established, where $x := v(X)$, $g := v(G)$, $G := 1/2(F + F')$ and $\frac{\partial g}{\partial x'}$ is the Magnus-Neudecker derivative matrix for the vector function $g(x)$.

The scalar Haffian can, of course, immediately be derived from the corresponding matrix Haffian as established in the present article.

**REFERENCES**


Haff, L.R. (1981). «Further identities for the Wishart distribution with applications in regression». *Canad. J. Statist.*, 9, 215-224.

Haff, L.R. (1982). «Identities for the inverse Wishart distribution with computational results in linear and quadratic discrimination». *Sankhyã* B, 44, 245-258.

Magnus, J.R. and Neudecker, H. (1999). *Matrix Differential Calculus with Applications in Statistics and Econometrics*, revised edition. John Wiley, Chichester.

Neudecker, H. (2000). *A note on the scalar Haffian*. *Qüestiió*, 24 (2), 243-249.