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THE PROPORTIONAL LIKELIHOOD RATIO ORDER AND APPLICATIONS

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In this paper, we introduce a new stochastic order between continuous non-negative random variables called the PLR (proportional likelihood ratio) order, which is closely related to the usual likelihood ratio order. The PLR order can be used to characterize random variables whose logarithms have log-concave (log-convex) densities. Many income random variables satisfy this property and they are said to have the IPLR (increasing proportional likelihood ratio) property (DPLR property). As an application, we show that the IPLR and DPLR properties are sufficient conditions for the Lorenz ordering of truncated distributions.

Keywords: Likelihood ratio order, proportional likelihood ratio order, ILR, DLR, IPLR, DPLR, log-concave density function, Lorenz order, truncated distributions

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1. INTRODUCTION

In this paper, we introduce the PLR (proportional likelihood ratio) order as a new stochastic order among continuous random variables, and two classes of probability distributions, the IPLR and DPLR classes, based on this order. Throughout this paper, the term *increasing* means *non-decreasing but not identically equal to a constant* and *decreasing* has an analogous meaning.

The PLR order is related to the likelihood ratio order, which is defined as follows (see, e.g., Ross, 1983).

Definition 1. *Let X and Y be continuous random variables with densities f and g, respectively, such that*

$$\frac{f(x)}{g(x)} \text{ decreases over the union of the supports of } X \text{ and } Y$$

(here $a/0$ is taken to be equal to ∞ whenever $a > 0$). Then X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{lr} Y$).

Many properties of the likelihood ratio order are listed in Section 1.C of Shaked and Shanthikumar (1994). The order $\leq_{lr}$ can be used to characterize random variables whose logarithms have log-concave (log-convex) densities (see Shaked and Shanthikumar,1994, Theorem 1.C.22). It can be shown that

$$(1) \qquad X + t \leq_{lr} X + t' \text{ for all } t \leq t' \iff \log\ f(x) \text{ is concave.}$$

The characterization (1) shows that log-concavity of densities can be interpreted as an ageing notion in reliability theory. In this sense, we have the following definition (see Ross, 1983).

Definition 2. *The continuous random variable X having density f is said to have the ILR (increasing likelihood ratio) property if* $\log\ f(x)$ *is concave and is said to have the DLR (decreasing likelihood ratio) property if* $\log\ f(x)$ *is convex.*

In Section 3 we introduce the IPLR (increasing proportional likelihood ratio) and DPLR (decreasing proportional likelihood ratio) classes. The basic properties of these classes are proven. In particular we show that the IPLR (DPLR) property can be used to characterize non-negative random variables whose logarithms have log-concave (log-convex) densities. A purpose of this Section is to find conditions under which a continuous random variable X is said to have the IPLR (DPLR) property.

In Section 4 we apply the IPLR and DPLR properties to comparisons of truncated random variables according to the Lorenz order. The Lorenz order is closely connected to the so-called Lorenz curve, defined as follows. Suppose $F(x)$ is the distribution function of a non-negative random variable X with finite mean μ. Let F^{-1} denote the inverse of F defined by

$$F^{-1}(p) = \inf\{x : F_X(x) \geq p\}, \quad p \in [0,1],$$

then the Lorenz curve corresponding to X can be defined (Gastwirth, 1971) as:

$$L_X(p) = \frac{1}{\mu}\int_0^p F^{-1}(t)dt \qquad 0 \leq p \leq 1. \tag{2}$$

The Lorenz curve is used in economics to measure the inequality of incomes. If X represents annual income, $L_X(p)$ is the proportion of total income that accrues to individuals having the $100p\%$ lowest incomes. The Lorenz curve provides the next partial ordering between random variables with finite means (see Arnold, 1987).

Definition 3. *We say that* $X \leq_L Y \Leftrightarrow L_X(p) \geq L_Y(p)$ *for every* $0 \leq p \leq 1$.

Let $S(f)$ be the number of sign changes of the function $f(x)$. The next theorem, from Arnold (1987), will be used in Section 4 to show that IPLR and DPLR properties are sufficient conditions to obtain orderings of the truncated distribution by the Lorenz order.

Theorem 1. *Let X and Y be two non-negative random variables with finite means μ_X and μ_Y, respectively, and let F and G be the corresponding distribution functions. If $S(F(x\mu_X) - G(x\mu_Y)) = 1$ and the sign sequence is $-,+$, then $X \leq_L Y$.*

2. THE PROPORTIONAL LIKELIHOOD RATIO ORDER

A new order closely related to the likelihood ratio order will next be described.

Definition 4. *Let X and Y be non-negative and absolutely continuous random variables with supports* $\mathrm{supp}(X)$ *and* $\mathrm{supp}(Y)$, *respectively. Denote the density functions of X and Y by f and g, respectively. Suppose that*

$$\frac{g(\lambda x)}{f(x)} \text{ increases in } x \text{ for any positive constant } \lambda < 1 \tag{3}$$

over the union of the supports of X and Y (here $a/0$ is taken to be equal to ∞ whenever $a>0$). Then, we say that X is smaller than Y in the proportional likelihood ratio order (denoted as $X \leq_{plr} Y$).

Example 1. *Let X_i be, $i=1,2$, the exponential random variable with parameter α_i. Its probability density function is $f_i(x)=\alpha_i \exp\{-\alpha_i x\}, x>0$. Then, it is easy to see that $X_2 \leq_{plr} X_1$ whenever $\alpha_1<\alpha_2$.*

A consequence of definition 4 is shown next. Suppose that X and Y are random variables whose supports are intervals with non-empty intersection, and let $l_X=\inf\{x : x \in \operatorname{supp}(X)\}$ and $u_X=\sup\{x : x \in \operatorname{supp}(X)\}$. Define l_Y and u_Y similarly.

Theorem 2. *If $X \leq_{plr} Y$, then $l_X \leq l_Y$ and $u_X \leq u_Y$.*

Proof. Suppose that $l_X>l_Y$. Let t_1,t_2 be such that $l_Y<t_1<l_X<t_2<\min\{u_X,u_Y\}$ and let $\lambda \in (0,1)$ such that $l_Y<\lambda t_1<l_X<\lambda t_2<\min\{u_X,u_Y\}$. Then $g(\lambda t_1)/f(t_1)=\infty> g(\lambda t_2)/f(t_2)$, in contradiction to (3). Therefore we must have $l_X \leq l_Y$. Similarly, it can be shown that $u_X \leq u_Y$. □

If X and Y are two random variables with respective supports (l_X,u_X) and (l_Y,u_Y) such that $l_X \leq l_Y$ and $u_X \leq u_Y$, it should be noted here that in (3) it is sufficient to consider only f and g such that $g(\lambda x)/f(x)$ increases in x over

$$\Lambda(\lambda)=\{x \in \operatorname{supp}(X) \text{ such that } \lambda x \in \operatorname{supp}(Y)\},$$

for all $\lambda \in (0,1)$ rather than over the union of the supports of X and Y.

The next result shows that the plr order has the property of ordering by size.

Theorem 3. *Let X and Y be non-negative and absolutely continuous random variables. If $X \leq_{plr} Y$, then $\mu_X \leq \mu_Y$.*

Proof. Let f and g be the density functions of X and Y, respectively, and for each $\lambda \in (0,1)$ let $g_{\frac{Y}{\lambda}}$ denote the density function of the random variable $\frac{Y}{\lambda}$. Suppose, by contradiction, that $\mu_X>\mu_Y$. Since

$$g(\lambda x)=\frac{1}{\lambda} g_{\frac{Y}{\lambda}}(x)$$

it follows from the assumptions that

(4) $$g_{\frac{Y}{\lambda}}(x)/f(x) \text{ is increasing in } x \text{ for all } \lambda \text{ in } (0,1).$$

Hence

$$S\left(g_{\frac{Y}{\lambda}} - f\right) = 1 \text{ for each } \lambda \in (0,1),$$

that is, X and $\frac{Y}{\lambda}$ are stochastically ordered for each λ in $(0,1)$. In particular, by taking

$$\lambda = \frac{\mu_Y}{\mu_X} < 1$$

it follows that the random variables X and $\frac{\mu_X}{\mu_Y}Y$ are stochastically ordered. Since X and $\frac{\mu_X}{\mu_Y}Y$ have the same mean, ordinary stochastic order is only possible if they have the same distribution. This contradicts (4) and hence $\mu_X \leq \mu_Y$ holds. □

The following result characterizes the proportional likelihood ratio order by means of the order $\leq_{lr}$.

Theorem 4. *The two absolutely continuous random variables X and Y satisfy $X \leq_{plr} Y$ if and only if $X \leq_{lr} aY$ for all $a > 1$.*

Proof. Note that

$$\frac{g_{aY}(x)}{f(x)} = \frac{g(x/a)}{af(x)} = \lambda\frac{g(\lambda x)}{f(x)}, \quad \lambda = \frac{1}{a} < 1$$

and the result holds. □

Theorem 5. *Let X and Y be non-negative and absolutely continuous random variables. Suppose that Y has a log-concave density function. Then*

(5) $$X \leq_{lr} Y \Longrightarrow X \leq_{plr} Y.$$

Proof. It is well known that if a non-negative random variable Y has a log-concave density and $a > 1$, then $Y \leq_{lr} aY$ (see, for example, Section 1.C. in Shaked and Shanthikumar, 1994). Since $X \leq_{lr} Y$ by assumption and the relation $\leq_{lr}$ is a transitive order, it follows that $X \leq_{lr} aY$. From Theorem 4 we obtain (5). □

3. INCREASING AND DECREASING PROPORTIONAL LIKELIHOOD RATIO

Definition 5. *Let X be a continuous non-negative random variable with density f. It will be said that X is increasing proportional likelihood ratio (IPLR) if*

$$\frac{f(\lambda x)}{f(x)} \text{ is increasing in } x \text{ for any positive constant } \lambda < 1 \tag{6}$$

It will be said that X is decreasing proportional likelihood ratio (DPLR) if

$$\frac{f(\lambda x)}{f(x)} \text{ is decreasing in } x \text{ for any positive constant } \lambda < 1.$$

(By convention, $\frac{a}{0} = +\infty$ whenever $a > 0$).

The study of the increase of $f(\lambda x)/f(x)$ can be restricted to the case of both arguments are in the support of X, as the next result shows. The proof is easy and is therefore omitted.

Theorem 6. *Let X be a continuous non-negative random variable with density f and suppose that the support of X is an interval. Then, X is IPLR if and only if*

$$\frac{f(\lambda x)}{f(x)} \text{ is increasing in } x \text{ over } \Lambda(\lambda) \text{ for all } \lambda \in (0,1)$$

where

$$\Lambda(\lambda) = \{x \in \mathrm{supp}(X) \text{ such that } \lambda x \in \mathrm{supp}(X)\}.$$

From Theorem 4 we have the following characterization of IPLR random variables in terms of the $\leq_{plr}$ and $\leq_{lr}$ orders. □

Theorem 7. *Let X be a non-negative and absolutely continuous random variable. The following conditions are equivalent:*

a) X is IPLR.
b) $X \leq_{lr} aX, \forall a > 1$.
c) $X \leq_{plr} X$.

Now, combining Theorem 7 with the argument used in the proof of Theorem 5, we obtain the following sufficient condition for the property IPLR.

Theorem 8. *Let X be a non-negative and absolutely continuous random variable with a log-concave density function. Then, X is IPLR.*

In other words, using Definition 2 we have that

$$X \text{ is ILR} \Longrightarrow X \text{ is IPLR}.$$

The class of IPLR (DPLR) random variables can be used to characterize random variables, whose logarithms have log-concave (log-convex) densities. This is shown in the following results.

Theorem 9. *Let X be an absolutely continuous non-negative random variable with density f. Then, X is IPLR (DPLR) if and only if $f(e^x)$ is log-concave (log-convex).*

Proof. We will prove the result for the IPLR case; the DPLR case can be proven in a similar way. Denote $g(x) = f(e^x)$ and suppose that $g(x)$ is log-concave, that is,

$$g(ax + (1-a)y) \geq g(x)^a g(y)^{1-a}, \ 0 \leq a \leq 1$$

for all x and y in the domain of g or, equivalently,

$$\frac{g(y_2 - x_2)\, g(y_1 - x_1)}{g(y_2 - x_1)\, g(y_1 - x_2)} \geq 1, \quad \forall x_1 < x_2, \forall y_1 < y_2. \tag{7}$$

Let $\lambda < 1$ and select t_1 and t_2 such that $0 \leq t_1 < t_2$. By taking $y_1 = \log\, t_1$, $y_2 = \log\, t_2$, $x_1 = 0$, $x_2 = -\log\, \lambda$ in (7), one obtains

$$\frac{f(\lambda t_1)}{f(t_1)} \leq \frac{f(\lambda t_2)}{f(t_2)},$$

that is, X is IPLR. Conversely, assume that X is IPLR and let $a > b$. Since

$$\frac{f(at)}{f(bt)} = \frac{f\left(\frac{a}{b} bt\right)}{f(bt)} = \frac{f(\lambda t')}{f(t')}, \ \ \lambda = a/b < 1, t' = bt,$$

the ratio $f(at)/f(bt)$ increases in t, that is,

$$\frac{f(at_1)}{f(bt_1)} \leq \frac{f(at_2)}{f(bt_2)}, \ \ \forall t_1 < t_2, \forall a < b. \tag{8}$$

Now, let $x_1 < x_2$ and $y_1 < y_2$. By taking $t_1 = e^{-x_2}$, $t_2 = e^{x_1}$, $a = e^{y_1}$, $b = e^{y_2}$ in (8) we obtain (7) and the result holds. □

The next result follows from Theorem 9. The proof is obvious and it is omitted.

Corollary 1. *Let X be an absolutely continuous non-negative random variable with density f. Then, X is IPLR (DPLR) if and only if $\log X$ has a log-concave (log-convex) density. Equivalently, using Definition 2, we can say that*

$$X \text{ is IPLR (DPLR)} \Longleftrightarrow \log X \text{ is ILR (DLR)}.$$

Let X and Y be two absolutely continuous non-negative random variables. If $X \leq_{lr} Y$, then it is not necessarily true that $X \leq_{plr} Y$. However, if one of these random variables is IPLR, then the relationship is verified (when X and Y have the same support $\mathbb{R}^+$). Since

$$\frac{f(\lambda x)}{g(x)} = \frac{f(\lambda x)}{f(x)}\frac{f(x)}{g(x)} = \frac{f(\lambda x)}{g(\lambda x)}\frac{g(\lambda x)}{g(x)}, \quad \forall x, \lambda x \in \mathbb{R}^+,$$

it is easy to prove the next result.

Theorem 10. *Let X and Y be two non-negative and absolutely continuous random variables having the same support $\mathbb{R}^+$. If $X \leq_{lr} Y$ and X or Y is IPLR, then $X \leq_{plr} Y$.*

The next result yields random variables with the IPLR property by means of a simple factorization of the density function.

Theorem 11. *Let X be a continuous non-negative random variable with finite mean, the support of which is an interval. If f, the density function of X, satisfies that*

$$f(\lambda x) = A(\lambda) \cdot B(x) \cdot \exp\{C(\lambda) \cdot D(x)\}, \quad \forall \lambda \tag{9}$$

whenever $x, \lambda x \in \operatorname{supp}(X)$, where:
$A(\lambda)$ and $C(\lambda)$ are independent of x,
$B(x)$ and $D(x)$ are independent of λ,
$C(\lambda)$ decreases in λ,
$D(x)$ increases in x,
then X is IPLR.

Proof. Let λ be a positive constant, $\lambda < 1$. Consider the ratio

$$h(x,\lambda) = \frac{f(\lambda x)}{f(x)} = \frac{A(\lambda)}{A(1)} \cdot \exp\{[C(\lambda) - C(1)]\, D(x)\} > 0$$

then

$$\frac{\partial}{\partial x}h(x,\lambda) = [C(\lambda) - C(1)] \cdot D'(x) \cdot h(x,\lambda) > 0$$

by assumption. It follows that $h(x,\lambda)$ is increasing, that is, X is IPLR. □

Remark 1. *Note that if $C(\lambda)$ increases in λ, then X is DPLR.*

In many distributions, (9) is very easy to verify. As an example, consider the three-parameter Amoroso distribution, with density function

$$f(x) = \frac{a^p}{|s|\,\Gamma(p)} x^{\frac{p}{s}-1} \exp\left\{-a x^{\frac{1}{s}}\right\}, \ x > 0, \ p > 0, \ a > 0, \ \frac{p}{s} \neq 0$$

Then

$$f(\lambda x) = \frac{a^p}{|s|\,\Gamma(p)} \cdot \lambda^{\frac{p}{s}-1} \cdot x^{\frac{p}{s}-1} \cdot \exp\left\{-a \lambda^{\frac{1}{s}} x^{\frac{1}{s}}\right\}.$$

By taking

$$A(\lambda) = \tfrac{a^p}{|s|\Gamma(p)} \lambda^{\frac{p}{s}-1}, \ B(x) = x^{\frac{p}{s}-1},$$

$$C(\lambda) = \begin{cases} -a\lambda^{\frac{1}{s}} & if \ s > 0 \\ a\lambda^{\frac{1}{s}} & if \ s < 0 \end{cases}$$

$$D(x) = \begin{cases} x^{\frac{1}{s}} & if \ s > 0 \\ -x^{\frac{1}{s}} & if \ s < 0 \end{cases}$$

it follows that $f(x)$ satisfies the property (9).

The Amoroso family includes the standard Gamma $(\lambda = 1, s = 1)$, March $(s = 1)$, Vinci $(s = -1)$, Weibull $(p = 1)$, Exponential $(p = 1, s = 1)$, Rayleigh $\left(p = 1, s = \frac{1}{2}\right)$, Chi-Square $\left(\lambda = \frac{1}{2}, p = \frac{n}{2}\right)$, Half-Normal $\left(\lambda = \frac{1}{2\sigma^2}, \ p = \frac{1}{2}, s = \frac{1}{2}\right)$, and Maxwell distributions $\left(p = \frac{3}{2}, s = \frac{1}{2}\right)$.

Similarly, the Dagum type I, Singh-Maddala, Generalized Beta of second kind, Three-Parameter Generalized Gamma, Log-Gomperz and Lognormal distributions have the property of IPLR. On the other hand, the random variable X having density function

$$f(x) = e^x, \ 0 < x < \log 2$$

is an example of DPLR distribution.

4. APPLICATIONS

Several authors have studied the effects of truncation of the random variable upon the Lorenz curve. Bhattacharya (1963) showed that under certain conditions on the support of the distribution, the Lorenz curve of a left truncated income distribution is independent of the point of truncation if, and only if, the incomes follow the Pareto law. For the right truncation case, Moothathu (1991) showed that the Lorenz curve is independent of the point of truncation if, and only if, incomes follow a power distribution. For random variables with absolutely continuous distributions, Ord *et al.* (1983) obtained an ordering in the Lorenz sense of the left truncated random variables, in terms of the mean residual life. Also for absolutely continuous random variables, Belzunce *et al.* (1995) gave some conditions in terms of the proportional failure rate and the elasticity of the random variable to obtain orderings of the truncated random variables by the Lorenz order.

Consider a continuous non-negative random variable X with distribution function F and survival function $\overline{F} = 1 - F$. The left truncated random variable of X in t is

$$X_{(t,\infty)} = \{X \mid X > t\}, \quad t \in \operatorname{supp}(X),$$

and the corresponding survival function is given by

$$\overline{F}_{(t,\infty)}(x) = \begin{cases} 1 & x < t \\ \frac{\overline{F}(x)}{\overline{F}(t)} & x \geq t. \end{cases}$$

The right truncated random variable of X in t is

$$X_{(-\infty,t)} = \{X \mid X < t\}, \quad t \in \operatorname{supp}(X),$$

whose survival function is given by

$$\overline{F}_{(-\infty,t)}(x) = \begin{cases} \frac{F(t)-F(x)}{F(t)} & x \leq t \\ 0 & x > t. \end{cases}$$

Before obtaining the main results of this section, we need to state the following definition.

Definition 6. *We say that X is an increasing failure rate (IFR) random variable if $\overline{F}$ is log-concave and we say that it is a decreasing failure rate (DFR) random variable if $\overline{F}$ is log-convex on its support.*

The IFR or DFR random variables are of interest in reliability theory. It can be shown (see, for example, Bryson and Siddiqui, 1969; Barlow and Proschan, 1975; Ross, 1983) that X is IFR (DFR) if and only if

(10) $$\frac{\overline{F}(x+c)}{\overline{F}(x)} \text{ is decreasing (increasing) in } x>0 \text{ for all } c\geq 0.$$

If we let $\widetilde{F}(x)=\overline{F}(e^x)$, it follows from (10) that the random variable $\log X$ is IFR (DFR) if and only if $\widetilde{F}(x+c)/\widetilde{F}(x)$ is decreasing (increasing) in x for all $c\geq 0$. Substituting $e^x=t$ it is seen that $\log X$ is IFR (DFR) if and only if

(11) $$\frac{\overline{F}(cx)}{\overline{F}(x)} \text{ is decreasing (increasing) in } x \in \operatorname{supp}(X) \text{ for all } c>1.$$

Theorem 12. *Let X be a non-negative continuous random variable. If the random variable $\log X$ is IFR (DFR) then $X_{(b,\infty)} \leq_L X_{(a,\infty)}$ $(\geq_L)$ for all $a<b,\ a,b\in \operatorname{supp}(X)$.*

Proof. We give the proof for the IFR case; the proof for the DFR case is similar.

Denote by $\mu_{(a,\infty)}$ the mean of $X_{(a,\infty)}$. It is easy to see that $\mu_{(a,\infty)}<\mu_{(b,\infty)}$ for all $a<b$ and since

$$\frac{\overline{F}\left(x\mu_{(b,\infty)}\right)}{\overline{F}\left(x\mu_{(a,\infty)}\right)} = \frac{\overline{F}\left[x\mu_{(a,\infty)}\left(\frac{\mu_{(b,\infty)}}{\mu_{(a,\infty)}}\right)\right]}{\overline{F}\left(x\mu_{(a,\infty)}\right)} =$$

$$= \frac{\overline{F}(yt)}{\overline{F}(y)},\ t=\frac{\mu_{(b,\infty)}}{\mu_{(a,\infty)}},\ y=x\mu_{(a,\infty)},$$

it follows that if $\log X$ is IFR then $\overline{F}\left(x\mu_{(b,\infty)}\right)/\overline{F}\left(x\mu_{(a,\infty)}\right)$ decreases in x for all $a<b$. Now consider the ratio

(12) $$r_{ab}(x)=\frac{\overline{F}_{(a,\infty)}\left(x\mu_{(a,\infty)}\right)}{\overline{F}_{(b,\infty)}\left(x\mu_{(b,\infty)}\right)},\ x>0,\ a<b,\ (a,b\in\operatorname{supp}(X))$$

where

$$\overline{F}_{(t,\infty)}(cx)=\begin{cases} 1 & \text{if } x<t/c \\ \overline{F}(cx)/\overline{F}(t) & \text{if } x\geq t/c \end{cases}$$

(in (12), $k/0$ is taken to be equal to ∞ whenever $k>0$). If $\log X$ is IFR then the ratio $\overline{F}\left(x\mu_{(a,\infty)}\right)/\overline{F}\left(x\mu_{(b,\infty)}\right)$ is increasing in x and we have that $r_{ab}(x)\geq 1$ for every x whenever $b/\mu_{(b,\infty)}\leq a/\mu_{(a,\infty)}$, and $S(r_{ab}(x)-1)\leq 1$ with the sign sequence being $-,+$, when equality holds whenever $b/\mu_{(b,\infty)}>a/\mu_{(a,\infty)}$. Therefore, $S(r_{ab}(x)-1)\leq 1$ with sign sequence $-,+$ in the case of equality. On the other hand,

$$\begin{aligned} S(r_{ab}(x)-1) &= S\left(\overline{F}_{(a,\infty)}\left(x\mu_{(a,\infty)}\right)-\overline{F}_{(b,\infty)}\left(x\mu_{(b,\infty)}\right)\right) \\ &= S\left(F_{(b,\infty)}\left(x\mu_{(b,\infty)}\right)-F_{(a,\infty)}\left(x\mu_{(a,\infty)}\right)\right) \text{ for all } a<b \end{aligned}$$

and from Theorem 1 it follows that $X_{(b,\infty)}\leq_L X_{(a,\infty)}$. □

Theorem 13. *Let X be a non-negative continuous random variable. If $F\left(e^{x}\right)$ is log-concave (log-convex) then $X_{(0,a)} \leq_L X_{(0,b)}$ $(\geq_L)$ for all $a < b,\ a,b \in \operatorname{supp}(X)$.*

Proof. Since $F\left(e^{x}\right)$ is log-concave (log-convex) if and only if

$$\frac{F(cx)}{F(x)} \text{ is decreasing (increasing) in } x > 0 \text{ for all } c \geq 0,$$

the result can be proven in the same way as the proof of Theorem 12. □

The previous results will allow us to show that the IPLR and DPLR properties (satisfied for many income distributions, as can be seen from Theorem 11) are sufficient conditions for the ordering of truncated distributions.

Corollary 2. *Let X be a non-negative and absolutely continuous random variable. If X is IPLR, then $X_{(a,\infty)} \geq_L X_{(b,\infty)}$ and $X_{(0,a)} \leq_L X_{(0,b)}$ for all $a < b$, $a,b \in \operatorname{supp}(X)$.*

Proof. If X is IPLR, it follows from Theorem 9 that $\log X$ has a log-concave density. It is well known (see, e.g., Prekopa, 1973) that the hypothesis of logconcavity of the density function implies the logconcavity of the distribution function and the survival function. Now, the result follows by applying Theorems 12 and 13. □

The following corollary can be proven in an analogous way to the previous.

Corollary 3. *Let X be a non-negative and absolutely continuous random variable. If X is DPLR, then $X_{(a,\infty)} \leq_L X_{(b,\infty)}$ and $X_{(0,a)} \geq_L X_{(0,b)}$ for all $a < b$, $a,b \in \operatorname{supp}(X)$.*

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