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Information matrices for some elliptically symmetric distributions

Saralees Nadarajah[1], Samuel Kotz[2]

[1] *University of Nebraska,* [2] *The George Washington University*

Abstract

The Fisher information matrices are derived for three of the most popular elliptically symmetric distributions: the Pearson type II, Pearson type VII and the Kotz type distributions. We hope the results could be important to the many researchers working in this area.

MSC: 33C90, 62E99

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1 Introduction

The elliptically symmetric Pearson type II, Pearson type VII and the Kotz type distributions are given by the joint pdfs

$$f(x, y) = \frac{N+1}{\pi \sqrt{1-\rho^2}} \left(1 - \frac{x^2 + y^2 - 2\rho xy}{1-\rho^2}\right)^N \tag{1}$$

(for $N > -1$ and $-1 < \rho < 1$),

* *Address for correspondence:* Saralees Nadarajah, Department of Statistics, University of Nebraska, Lincoln, NE 68583.
Samuel Kotz, Department of Engineering Management and Systems Engineering, The George Washington University, Washington, D.C. 20052.
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$$f(x,y) = \frac{N-1}{\pi m \sqrt{1-\rho^2}} \left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)} \right)^{-N} \quad (2)$$

(for $N > 1$, $m > 0$ and $-1 < \rho < 1$), and

$$f(x,y) = \frac{s r^{N/s} \left(x^2+y^2-2\rho xy\right)^{N-1}}{\pi \Gamma(N/s) \left(1-\rho^2\right)^{N-1/2}} \exp\left\{ -r \left(\frac{x^2+y^2-2\rho xy}{1-\rho^2} \right)^s \right\} \quad (3)$$

(for $N > 0$, $r > 0$, $s > 0$ and $-1 < \rho < 1$), respectively. The bivariate t-distribution and the bivariate Cauchy distribution are special cases of (2) for $N = (m+2)/2$ and $m = 1$, $N = 3/2$, respectively. When $s = 1$, (3) is the original Kotz distribution introduced in Kotz (1975). When $N = 1$, $s = 1$ and $r = 1/2$, (3) reduces to a bivariate normal density. The parameter ρ is the correlation coefficient between the x and y components. For details on properties and applications of these distributions see Johnson (1987), Fang *et al.* (1990), Nadarajah (2003) and Kotz and Nadarajah (2004).

The aim of this note is to calculate the Fisher information matrices corresponding to each of the pdfs given by (1), (2) and (3). This requires calculation of product moments of the form $E(X^m Y^n)$. A transformation which aides this task is:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{1+\rho} + \sqrt{1-\rho} & \sqrt{1+\rho} - \sqrt{1-\rho} \\ \sqrt{1+\rho} - \sqrt{1-\rho} & \sqrt{1+\rho} + \sqrt{1-\rho} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}. \quad (4)$$

Under this transformation, one can easily see that (1), (2) and (3) reduce to

$$f(u,v) = \frac{N+1}{\pi} \left(1-u^2-v^2\right)^N, \quad (5)$$

$$f(x,y) = \frac{N-1}{\pi m} \left(1 + \frac{u^2+v^2}{m}\right)^{-N}, \quad (6)$$

and

$$f(u,v) = \frac{s r^{N/s} \left(x^2+y^2\right)^{N-1}}{\pi \Gamma(N/s)} \exp\left\{ -r \left(x^2+y^2\right)^s \right\}, \quad (7)$$

respectively. Furthermore, the product moments for (5), (6) and (3) are given by

$$E\left(U^p V^q\right) = \frac{N+1}{\pi} B\left(N+1, \frac{p+q}{2}+1\right) B\left(\frac{p+1}{2}, \frac{q+1}{2}\right), \quad (8)$$

$$E\left(U^p V^q\right) = \frac{m^{(p+q)/2}(N-1)}{\pi} B\left(\frac{p+q}{2}+1, N-\frac{p+q}{2}-1\right) B\left(\frac{p+1}{2}, \frac{q+1}{2}\right), \quad (9)$$

and

$$E\left(U^p V^q\right) = \frac{\Gamma\left(\frac{2N+p+q}{2s}\right)}{\pi r^{(p+q)/(2s)} \Gamma\left(N/s\right)} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right), \quad (10)$$

respectively, if both $p \geq 0$ and $q \geq 0$ are even integers (see, for example, Fang *et al.* (1990)). If either p or q is odd then the moment equals to zero. Hence, the product moment $E(X^m Y^n)$ for any m and n can be calculated by combining (4), (8), (9) and (10). Certain relationships which express $E(X^m Y^n)$ in terms of product moments of (U, V) are given in the Appendix (Section 5).

The calculations for the Kotz type distribution require additional moments of the form

$$J(m,n) = E\left[\left(\frac{X^2+Y^2-2\rho XY}{1-\rho^2}\right)^m \left\{\log\left(\frac{X^2+Y^2-2\rho XY}{1-\rho^2}\right)\right\}^n\right] \quad (11)$$

and

$$K(m,n) = E\left[XY\left(\frac{X^2+Y^2-2\rho XY}{1-\rho^2}\right)^m \left\{\log\left(\frac{X^2+Y^2-2\rho XY}{1-\rho^2}\right)\right\}^n\right]. \quad (12)$$

For this, apply the transformation

$$(X, Z) = \left(X, \frac{X^2+Y^2-2\rho XY}{1-\rho^2}\right).$$

Note that the jacobian of this transformation is:

$$\mid J \mid = \frac{\sqrt{1-\rho^2}}{2\sqrt{Z-X^2}}.$$

Thus, (11) and (12) can be reduced to

$$J(m,n) = \frac{2sr^{N/s} P(m,n)}{\pi\Gamma\left(N/s\right)\left(1-\rho^2\right)^{N-1}} \quad (13)$$

and

$$K(m,n) = \frac{2sr^{N/s} \rho Q(m,n)}{\pi\Gamma\left(N/s\right)\left(1-\rho^2\right)^{N-1}}, \quad (14)$$

respectively, where $P(m,n)$ and $Q(m,n)$ denote the integrals

$$P(m,n) = \int_0^\infty \int_0^{\sqrt{z}} \frac{z^{N+m-1}(\log z)^n \exp(-rz^s)}{\sqrt{z-x^2}} dxdz$$

and

$$Q(m,n) = \int_0^\infty \int_0^{\sqrt{z}} \frac{x^2 z^{N+m-1}(\log z)^n \exp(-rz^s)}{\sqrt{z-x^2}} dxdz.$$

Integrating with respect to the x variable yields

$$P(m,n) = \frac{\pi}{2} \int_0^\infty z^{N+m-1}(\log z)^n \exp(-rz^s)\, dz$$

and

$$Q(m,n) = \frac{\pi}{4} \int_0^\infty z^{N+m}(\log z)^n \exp(-rz^s)\, dz.$$

These integrals can be calculated by using equation (2.6.21.1) in Prudnikov *et al.* (1986, volume 1) to yield

$$P(m,n) = \frac{\pi}{2s} \left(\frac{\partial}{\partial \alpha}\right)^n \left[r^{-\alpha/s} \Gamma\left(\frac{\alpha}{s}\right) \right] \Bigg|_{\alpha=N+m} \tag{15}$$

and

$$Q(m,n) = \frac{\pi}{4s} \left(\frac{\partial}{\partial \alpha}\right)^n \left[r^{-\alpha/s} \Gamma\left(\frac{\alpha}{s}\right) \right] \Bigg|_{\alpha=N+m+1}. \tag{16}$$

Hence, (11) and (12) can be calculated by combining (13)–(14) and (15)–(16).

The exact forms of the information matrices are given in Sections 2, 3 and 4. The calculations use the digamma function defined by $\Psi(x) = d \log \Gamma(x)/dx$.

2 Information matrix for Pearson II

If (x, y) is a single observation from (1) then the log-likelihood function can be written as

$$\log L(N,\rho) = \log(N+1) - \log \pi - \frac{1}{2} \log\left(1-\rho^2\right) + N \log\left(1 - \frac{x^2+y^2-2\rho xy}{1-\rho^2}\right).$$

The first-order derivatives are:

$$\frac{\partial \log L}{\partial N} = \frac{1}{N+1} + \log\left(1 - \frac{x^2 + y^2 - 2\rho xy}{1-\rho^2}\right)$$

and

$$\frac{\partial \log L}{\partial \rho} = \frac{\rho}{1-\rho^2} - \frac{2N\left\{\rho x^2 + \rho y^2 - \left(1+\rho^2\right) xy\right\}}{\left(1-\rho^2\right)^2}\left(1 - \frac{x^2 + y^2 - 2\rho xy}{1-\rho^2}\right)^{-1}.$$

The second-order derivatives are:

$$\frac{\partial^2 \log L}{\partial N^2} = -\frac{1}{(N+1)^2},$$

$$\frac{\partial^2 \log L}{\partial N \partial \rho} = -\frac{2\left\{\rho x^2 + \rho y^2 - \left(1+\rho^2\right) xy\right\}}{\left(1-\rho^2\right)^2}\left(1 - \frac{x^2 + y^2 - 2\rho xy}{1-\rho^2}\right)^{-1},$$

and

$$\frac{\partial^2 \log L}{\partial \rho^2} = \frac{1+\rho^2}{\left(1-\rho^2\right)^2} - \frac{4N\left\{\rho x^2 + \rho y^2 - \left(1+\rho^2\right) xy\right\}^2}{\left(1-\rho^2\right)^4}\left(1 - \frac{x^2 + y^2 - 2\rho xy}{1-\rho^2}\right)^{-2}$$

$$- \frac{2N\left[\left(1-\rho^2\right)\left(x^2 + y^2 - 2\rho xy\right) + 4\rho\left\{\rho x^2 + \rho y^2 - \left(1+\rho^2\right) xy\right\}\right]}{\left(1-\rho^2\right)^3}.$$

Now, we can compute the elements of the Fisher information matrix. It is clear that

$$E\left(-\frac{\partial^2 \log L}{\partial N^2}\right) = \frac{1}{(N+1)^2}.$$

By applying (8) and (17)–(19), one gets

$$E\left(-\frac{\partial^2 \log L}{\partial N \partial \rho}\right) = \frac{(N+1)B(2,N)\rho}{1-\rho^2}.$$

Finally, application of (8) and (17)–(24) yields

$$E\left(-\frac{\partial^2 \log L}{\partial \rho^2}\right) = \frac{1}{2\left(1-\rho^2\right)^2}\Big\{2N(N+1)B(3,N-1)\rho^2 + 4N(N+1)B(2,N)\rho^2 - 2\rho^2 + N(N+1)B(3,N-1) + 4N(N+1)B(2,N) - 2\Big\}.$$

3 Information matrix for Pearson VII

If (x, y) is a single observation from (2) then the log-likelihood function can be written as

$$\log L(N, m, \rho) = \log(N-1) - \log\pi - \log m - \frac{1}{2}\log\left(1-\rho^2\right) - N\log\left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right).$$

The first-order derivatives are:

$$\frac{\partial \log L}{\partial N} = \frac{1}{N-1} - \log\left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right),$$

$$\frac{\partial \log L}{\partial m} = \frac{N\left(x^2+y^2-2\rho xy\right)}{m^2\left(1-\rho^2\right)}\left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right)^{-1} - \frac{1}{m},$$

and

$$\frac{\partial \log L}{\partial \rho} = \frac{\rho}{1-\rho^2} - \frac{2N\left\{\rho x^2 + \rho y^2 - \left(1+\rho^2\right)xy\right\}}{m\left(1-\rho^2\right)^2}\left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right)^{-1}.$$

The second-order derivatives are:

$$\frac{\partial^2 \log L}{\partial N^2} = -\frac{1}{(N-1)^2},$$

$$\frac{\partial^2 \log L}{\partial N \partial m} = \frac{x^2+y^2-2\rho xy}{m^2\left(1-\rho^2\right)}\left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right)^{-1},$$

$$\frac{\partial^2 \log L}{\partial N \partial \rho} = -\frac{2\left\{\rho x^2 + \rho y^2 - \left(1+\rho^2\right) xy\right\}}{\left(1-\rho^2\right)^2} \left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right)^{-1},$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial m^2} &= \frac{1}{m^2} - \frac{2N\left(x^2+y^2-2\rho xy\right)}{m^3\left(1-\rho^2\right)} \left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right)^{-1} \\ &\quad + \frac{N\left(x^2+y^2-2\rho xy\right)^2}{m^4\left(1-\rho^2\right)^2} \left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right)^{-2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial m \partial \rho} &= \frac{2N\left\{\rho x^2 + \rho y^2 - \left(1+\rho^2\right) xy\right\}}{m^2\left(1-\rho^2\right)^2} \left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right)^{-1} \\ &\quad - \frac{2N\left(x^2+y^2-2\rho xy\right)\left\{\rho x^2 + \rho y^2 - \left(1+\rho^2\right) xy\right\}}{m^3\left(1-\rho^2\right)^3} \left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right)^{-2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 \log L}{\partial \rho^2} &= \frac{1+\rho^2}{\left(1-\rho^2\right)^2} + \frac{2N\left[\left(1-\rho^2\right)\left(x^2+y^2-2\rho xy\right) + 4\rho\left\{\rho x^2 + \rho y^2 - \left(1+\rho^2\right) xy\right\}\right]}{m^2\left(1-\rho^2\right)^3} \\ &\quad \times \left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right)^{-1} \\ &\quad \times \left[4\left\{\rho x^2 + \rho y^2 - \left(1+\rho^2\right) xy\right\}^2 \left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right)^{-1} - m\right].\end{aligned}$$

Now, we can compute the elements of the Fisher information matrix. It is clear that

$$E\left(-\frac{\partial^2 \log L}{\partial N^2}\right) = \frac{1}{(N-1)^2}.$$

By applying (9) and (17)–(19), one gets

$$E\left(-\frac{\partial^2 \log L}{\partial N \partial m}\right) = -\frac{(N-1)B(2, N-1)}{m}$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial N \partial \rho}\right) = \frac{(N-1)B(2, N-1)m\rho}{1-\rho^2}.$$

By applying (9) and (17)–(24), one gets

$$E\left(-\frac{\partial^2 \log L}{\partial m^2}\right) = \frac{2N(N-1)B(2, N-1) + N(1-N)B(3, N-1) - 1}{m^2}$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial m \partial \rho}\right) = \frac{N(N-1)\{B(3, N-1) - B(2, N-1)\}\rho}{m\left(1-\rho^2\right)}.$$

Finally, application of (9), (17)–(19) and (25)–(31) yields

$$\begin{aligned} E\left(-\frac{\partial^2 \log L}{\partial \rho^2}\right) = & \frac{1}{\left(1-\rho^2\right)^2}\Big\{2N(1-N)B(4, N-2)m\rho^8 + 3N(1-N)B(4, N-2)m\rho^6 \\ & +11N(N-1)B(4, N-2)m\rho^4 + 5N(1-N)B(4, N-2)m\rho^2 \\ & +2N(N-1)B(2, N-1)\rho^2 - \rho^2 + 2N(N-1)B(2, N-1) \\ & +N(1-N)B(4, N-2)m - 1\Big\}. \end{aligned}$$

4 Information matrix for Kotz type

If (x, y) is a single observation from (3) then the log-likelihood function can be written as

$$\begin{aligned} \log L(N, r, s, \rho) = & \log s + \frac{N \log r}{s} - \log \pi - \log \Gamma\left(\frac{N}{s}\right) + \left(\frac{1}{2} - N\right) \log\left(1-\rho^2\right) \\ & +(N-1)\log\left(x^2 + y^2 - 2\rho xy\right) - r\left(\frac{x^2 + y^2 - 2\rho xy}{1-\rho^2}\right)^s. \end{aligned}$$

The first-order derivatives are:

$$\frac{\partial \log L}{\partial N} = \frac{\log r}{s} - \frac{1}{s}\Psi\left(\frac{N}{s}\right) - \log\left(1-\rho^2\right) + \log\left(x^2 + y^2 - 2\rho xy\right),$$

$$\frac{\partial \log L}{\partial r} = \frac{N}{rs} - \left(\frac{x^2 + y^2 - 2\rho xy}{1 - \rho^2}\right)^s,$$

$$\frac{\partial \log L}{\partial s} = \frac{1}{s} - \frac{N \log r}{s^2} + \frac{N}{s^2}\Psi\left(\frac{N}{s}\right) - r\left(\frac{x^2 + y^2 - 2\rho xy}{1 - \rho^2}\right)^s \log\left(\frac{x^2 + y^2 - 2\rho xy}{1 - \rho^2}\right)$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \rho} &= \frac{(2N-1)\rho}{1-\rho^2} - \frac{2(N-1)xy}{x^2 + y^2 - 2\rho xy} \\ &\quad - \frac{2rs\left\{\rho x^2 + \rho y^2 - \left(1 + \rho^2\right)xy\right\}}{\left(1 - \rho^2\right)^2}\left(\frac{x^2 + y^2 - 2\rho xy}{1 - \rho^2}\right)^{s-1}.\end{aligned}$$

The second-order derivatives are:

$$\frac{\partial^2 \log L}{\partial N^2} = -\frac{1}{s^2}\Psi'\left(\frac{N}{s}\right),$$

$$\frac{\partial^2 \log L}{\partial N \partial r} = \frac{1}{rs},$$

$$\frac{\partial^2 \log L}{\partial N \partial s} = -\frac{\log r}{s^2} + \frac{1}{s^2}\Psi\left(\frac{N}{s}\right) + \frac{N}{s^3}\Psi'\left(\frac{N}{s}\right),$$

$$\frac{\partial^2 \log L}{\partial N \partial \rho} = \frac{2\rho}{1-\rho^2} - \frac{2xy}{x^2 + y^2 - 2\rho xy},$$

$$\frac{\partial^2 \log L}{\partial r^2} = -\frac{N}{r^2 s},$$

$$\frac{\partial^2 \log L}{\partial r \partial s} = -\frac{N}{rs^2} - \left(\frac{x^2 + y^2 - 2\rho xy}{1 - \rho^2}\right)^s \log\left(\frac{x^2 + y^2 - 2\rho xy}{1 - \rho^2}\right),$$

$$\frac{\partial^2 \log L}{\partial r \partial \rho} = -\frac{2s\left\{\rho x^2 + \rho y^2 - \left(1 + \rho^2\right)xy\right\}}{\left(1 - \rho^2\right)^2}\left(\frac{x^2 + y^2 - 2\rho xy}{1 - \rho^2}\right)^{s-1},$$

$$\frac{\partial^2 \log L}{\partial s^2} = -\frac{1}{s^2} + \frac{2N \log r}{s^3} - \frac{2N}{s^3}\Psi\left(\frac{N}{s}\right) - \frac{N^2}{s^4}\Psi'\left(\frac{N}{s}\right) -r\left\{\log\left(\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right)\right\}^2\left(\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right)^s,$$

$$\frac{\partial^2 \log L}{\partial s \partial \rho} = -\frac{2r\left\{\rho x^2+\rho y^2-\left(1+\rho^2\right)xy\right\}}{\left(1-\rho^2\right)^2}\left(\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right)^{s-1} \times\left\{1+s\log\left(\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right)\right\},$$

and

$$\frac{\partial^2 \log L}{\partial \rho^2} = \frac{(2N-1)\left(1+\rho^2\right)}{\left(1-\rho^2\right)^2} - \frac{4(N-1)x^2y^2}{\left(x^2+y^2-2\rho xy\right)^2} - \frac{4rs(s-1)\left\{\rho x^2+\rho y^2-\left(1+\rho^2\right)xy\right\}^2}{\left(1-\rho^2\right)^4} \times\left(\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right)^{s-2} - \frac{2rs\left[\left(1-\rho^2\right)\left(x^2+y^2-2\rho xy\right)+4\rho\left\{\rho x^2+\rho y^2-\left(1+\rho^2\right)xy\right\}\right]}{\left(1-\rho^2\right)^3} \times\left(\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right)^{s-1}.$$

Now, we can compute the elements of the Fisher information matrix. It is clear that

$$E\left(-\frac{\partial^2 \log L}{\partial N^2}\right) = \frac{1}{s^2}\Psi'\left(\frac{N}{s}\right).$$

$$E\left(-\frac{\partial^2 \log L}{\partial N \partial r}\right) = -\frac{1}{rs},$$

$$E\left(-\frac{\partial^2 \log L}{\partial N \partial s}\right) = \frac{\log r}{s^2} - \frac{1}{s^2}\Psi\left(\frac{N}{s}\right) - \frac{N}{s^3}\Psi'\left(\frac{N}{s}\right),$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial r^2}\right) = \frac{N}{r^2 s}.$$

By applying (10) and (19), one gets

$$E\left(-\frac{\partial^2 \log L}{\partial N \partial \rho}\right) = -\frac{\rho}{1-\rho^2}.$$

By applying (10) and (17)–(19), one gets

$$E\left(-\frac{\partial^2 \log L}{\partial r \partial \rho}\right) = \frac{N\rho}{r\left(1-\rho^2\right)}.$$

By applying (10) and (17)–(24), one gets

$$E\left(-\frac{\partial^2 \log L}{\partial \rho^2}\right) = \frac{1 + Ns\left(1+2\rho^2\right)}{2\left(1-\rho^2\right)^2}.$$

By applying (13), one gets

$$E\left(-\frac{\partial^2 \log L}{\partial r \partial s}\right) = \frac{N}{rs^2} + \frac{N\left\{\Psi\left(1+\frac{N}{s}\right) - \log r\right\}}{rs^2\left(1-\rho^2\right)^{N-1}}$$

and

$$\begin{aligned} E\left(-\frac{\partial^2 \log L}{\partial s^2}\right) = {} & \frac{1}{s^2} - \frac{2N\log r}{s^3} + \frac{2N}{s^3}\Psi\left(\frac{N}{s}\right) + \frac{N^2}{s^4}\Psi'\left(\frac{N}{s}\right) \\ & + \frac{N\left\{\Psi'\left(1+\frac{N}{s}\right) + \Psi^2\left(1+\frac{N}{s}\right) - 2\log r\Psi\left(1+\frac{N}{s}\right) + (\log r)^2\right\}}{s^3\left(1-\rho^2\right)^{N-1}}. \end{aligned}$$

Finally, application of (17)–(19), (13) and (14) yields

$$E\left(-\frac{\partial^2 \log L}{\partial s \partial \rho}\right) = \frac{N\rho}{s\left(1-\rho^2\right)} + \frac{2N\rho\left\{\Psi\left(1+\frac{N}{s}\right) - \log r\right\}}{s\left(1-\rho^2\right)^{N}} - \frac{N\rho\left\{\Psi\left(1+\frac{N}{s}\right) - \log r\right\}}{s\left(1-\rho^2\right)^{N+1}}.$$

5 Appendix

The following relationships are needed for the calculation of the elements of the Fisher information matrices. These relations follow directly from the transformation (4).

$$E\left(X^2\right) = \frac{1+\sqrt{1-\rho^2}}{2}E\left(U^2\right)+\frac{1-\sqrt{1-\rho^2}}{2}E\left(V^2\right), \tag{17}$$

$$E\left(Y^2\right) = \frac{1-\sqrt{1-\rho^2}}{2}E\left(U^2\right)+\frac{1+\sqrt{1-\rho^2}}{2}E\left(V^2\right), \tag{18}$$

$$E\left(XY\right) = \frac{\rho}{2}\left\{E\left(U^2\right)+E\left(V^2\right)\right\}, \tag{19}$$

$$E\left(X^4\right) = \frac{2\sqrt{1-\rho^2}+2-\rho^2}{4}E\left(U^4\right)-\frac{2\sqrt{1-\rho^2}-2+\rho^2}{4}E\left(V^4\right)+\frac{3\rho^2}{2}E\left(U^2V^2\right), \tag{20}$$

$$E\left(Y^4\right) = \frac{2\sqrt{1-\rho^2}+2-\rho^2}{4}E\left(V^4\right)-\frac{2\sqrt{1-\rho^2}-2+\rho^2}{4}E\left(U^4\right)+\frac{3\rho^2}{2}E\left(U^2V^2\right), \tag{21}$$

$$E\left(XY^3\right) = \frac{\rho\left\{1-\sqrt{1-\rho^2}\right\}}{4}E\left(U^4\right)+\frac{\rho\left\{1+\sqrt{1-\rho^2}\right\}}{4}E\left(V^4\right)+\frac{3\rho}{2}E\left(U^2V^2\right), \tag{22}$$

$$E\left(X^3Y\right) = \frac{\rho\left\{1-\sqrt{1-\rho^2}\right\}}{4}E\left(V^4\right)+\frac{\rho\left\{1+\sqrt{1-\rho^2}\right\}}{4}E\left(U^4\right)+\frac{3\rho}{2}E\left(U^2V^2\right), \tag{23}$$

$$E\left(X^2Y^2\right) = \frac{\rho^2}{4}\left\{E\left(U^4\right)+E\left(V^4\right)\right\}+\left(1+\frac{\rho^2}{2}\right)E\left(U^2V^2\right), \tag{24}$$

$$\begin{aligned} E\left(X^6\right) = {} & \frac{4-3\rho^2+\left(4-\rho^2\right)\sqrt{1-\rho^2}}{8}E\left(U^6\right)+\frac{4-3\rho^2-\left(4-\rho^2\right)\sqrt{1-\rho^2}}{8}E\left(V^6\right) \\ & +\frac{15\rho^2\left\{1-\sqrt{1-\rho^2}\right\}}{8}E\left(U^2V^4\right)+\frac{15\rho^2\left\{1+\sqrt{1-\rho^2}\right\}}{8}E\left(U^4V^2\right), \end{aligned} \tag{25}$$

$$E\left(Y^6\right) = \frac{4-3\rho^2+\left(4-\rho^2\right)\sqrt{1-\rho^2}}{8} E\left(V^6\right) + \frac{4-3\rho^2-\left(4-\rho^2\right)\sqrt{1-\rho^2}}{8} E\left(U^6\right) + \frac{15\rho^2\left\{1-\sqrt{1-\rho^2}\right\}}{8} E\left(U^4V^2\right) + \frac{15\rho^2\left\{1+\sqrt{1-\rho^2}\right\}}{8} E\left(U^2V^4\right), \quad (26)$$

$$\begin{aligned} E\left(XY^5\right) &= \frac{\rho\left\{2-\rho^2-2\sqrt{1-\rho^2}\right\}}{8} E\left(U^6\right) + \frac{\rho\left\{2-\rho^2+2\sqrt{1-\rho^2}\right\}}{8} E\left(V^6\right) \\ &+ \frac{5\rho\left\{2+\rho^2+2\sqrt{1-\rho^2}\right\}}{8} E\left(U^2V^4\right) \\ &+ \frac{5\rho\left\{2+\rho^2-2\sqrt{1-\rho^2}\right\}}{8} E\left(U^4V^2\right), \end{aligned} \quad (27)$$

$$\begin{aligned} E\left(X^5Y\right) &= \frac{\rho\left\{2-\rho^2-2\sqrt{1-\rho^2}\right\}}{8} E\left(V^6\right) + \frac{\rho\left\{2-\rho^2+2\sqrt{1-\rho^2}\right\}}{8} E\left(U^6\right) \\ &+ \frac{5\rho\left\{2+\rho^2+2\sqrt{1-\rho^2}\right\}}{8} E\left(U^4V^2\right) \\ &+ \frac{5\rho\left\{2+\rho^2-2\sqrt{1-\rho^2}\right\}}{8} E\left(U^2V^4\right), \end{aligned} \quad (28)$$

$$\begin{aligned} E\left(X^2Y^4\right) &= \frac{\rho^2\left\{1-\sqrt{1-\rho^2}\right\}}{8} E\left(U^6\right) + \frac{\rho^2\left\{1+\sqrt{1-\rho^2}\right\}}{8} E\left(V^6\right) \\ &+ \frac{4+11\rho^2+\left(4+\rho^2\right)\sqrt{1-\rho^2}}{8} E\left(U^2V^4\right) \\ &+ \frac{4+11\rho^2-\left(4+\rho^2\right)\sqrt{1-\rho^2}}{8} E\left(U^4V^2\right), \end{aligned} \quad (29)$$

$$
\begin{aligned}
E\left(X^4 Y^2\right) &= \frac{\rho^2 \left\{1 - \sqrt{1-\rho^2}\right\}}{8} E\left(V^6\right) + \frac{\rho^2 \left\{1 + \sqrt{1-\rho^2}\right\}}{8} E\left(U^6\right) \\
&\quad + \frac{4 + 11\rho^2 + \left(4+\rho^2\right)\sqrt{1-\rho^2}}{8} E\left(U^4 V^2\right) \\
&\quad + \frac{4 + 11\rho^2 - \left(4+\rho^2\right)\sqrt{1-\rho^2}}{8} E\left(U^2 V^4\right), \qquad (30)
\end{aligned}
$$

and

$$
E\left(X^3 Y^3\right) = \frac{\rho^3}{8}\left\{E\left(U^6\right) + E\left(V^6\right)\right\} + \frac{3\rho\left(4+\rho^2\right)}{8}\left\{E\left(U^2 V^4\right) + E\left(U^4 V^2\right)\right\}. \qquad (31)
$$

7 References

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